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  - [2] H. Quastler, ed. (1955), *Information Theory in Psychology,* The Free Press, Glencoe.
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Algebras and Logics Emerging Out of Rough Sets
(An Invited Review Paper)

Mihir Kumar Chakraborty and Masiur Rahaman Sardar

Abstract

This is basically a survey work though some new results have also been incorporated. The contents presented are from the research done by Chakraborty and his co-workers for last three decades. A major part consists in presentation of set-models of abstract algebraic structures generated from rough sets. Three Types of logics developed during the course of research are categorized. Some foundational issues are raised at the end and open questions are mentioned.

Keywords: Quasi-Boolean algebra, Pre-rough algebra, Hilbert system, Modal logic, Rough sets.

1. INTRODUCTION

Rough set theory was invented by Pawlak in the year 1982 [26] from the angle of computer-applications. But the theory has surpassed the boundary and entered the domains of mathematics, philosophy etc. In this paper we present a survey of the mathematical (algebraic and logical) work done by Chakraborty and his co-workers and collaborators in the theoretical domain.

According to the Pawlak's first paper [26], the universe U (a non empty set) is partitioned into equivalence classes by an attribute-value data table. For our purpose, the starting point is the pair (U, R), called the approximation space where U is the universe and R is an equivalence relation generating a partition. Any subset P of U is then approximated by two sets $\underline{P}_R$ and $\overline{P}^R$ called the lower and upper approximations of P and defined as follows:

$$\underline{P}_R = \left\{u \in U : [u]_R \subseteq P\right\}$$

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and

$$\overline{P}^R = \{u \in U : [u]_R \cap P \neq \emptyset\}$$

where $[u]_R$ is the equivalence class due to R to which u belongs. In the power set 2^U of U an equivalence relation $\approxeq$ is naturally generated by

$$P \approxeq Q \text{ if and only if } \underline{P}_R = \underline{Q}_R \text{ and } \overline{P}^R = \overline{Q}^R$$

in which case P and Q are called roughly equal. Any element in the quotient set $2^U/\approxeq$, $[P]_\approxeq$ has been defined as a rough set in [26]. There are equivalent definitions too (see [2]). Algebras generated in $2^U/\approxeq$ may be considered as the beginning of the abstract algebraic studies in rough sets. The relation R has been taken to be an arbitrary one in later years and instead of a relation covering has been imposed on U. In both the cases lower and upper approximations are defined and their properties are studied.

Side by side the algebras corresponding logical systems have been designed. This is done in three different ways which will be clarified later. However, it is necessary to be introduced to the basic notions of logical systems.

A logical system is the pair $(\mathcal{L}, \vdash)$ where $\mathcal{L}$ is a set of well-formed formulas (wffs) over an alphabet $\mathcal{A}$ of symbols and $\vdash$ a binary relation from $2^{\mathcal{L}}$, the power set of $\mathcal{L}$ to $\mathcal{L}$. In case of standard proposition logic the set $\mathcal{A} = \{p_1, p_2, p_3 ..., \wedge, \vee, \Rightarrow, \neg,), (\}$ and $\mathcal{L}$ is the set of finite strings on $\mathcal{A}$ given by:

$$p_i \,|\, \neg\alpha \,|\, \alpha \wedge \beta \,|\, \alpha \vee \beta \,|\, \alpha \Rightarrow \beta.$$

Usually $\neg$ and one of the binary connectives $\wedge$, $\vee$ and $\Rightarrow$ are taken as basic and the other two are defined (see [24]). A subset of $\mathcal{L}$ is taken as axiom set and Modus Ponens (**MP**) is the only rule of inference given by: ‘to derive β from α and $\alpha \Rightarrow \beta$’. With the help of axioms and MP the relation $\vdash$ (consequence relation) is defined (see [24]). Semantics is given by a valuation v which is a special mapping from $\mathcal{L}$ to B (an arbitrary Boolean Algebra). It is proved that $\Gamma \vdash \alpha$ holds if and only if $v(\Gamma) = \{1\}$ implies $v(\alpha) = 1$ where 1 is the greatest element of B.

Classical proposition logic is extended in Modal logic by first extending the alphabet with one unary operator L (and defining another operator M by $\neg L \neg$) and enhancing the set of axioms by modal axioms. Depending on the axioms the hierarchy of modal systems is constituted (see Section 3). Besides MP, another rule Necessitation (N) is taken and the consequence relation $\vdash_S$ for modal system S is defined (see [18]). Standard semantics of modal system is given in terms of Kripke frame (see [18]). For this paper a little detail of Kripke frame will be presented in Section 3. The aim here is to show that modal systems can be given rough set semantics and in the other direction some new modal systems are created from the existing rough set (covering based) models.

Section-wise details of this paper are as follows: Section 2 contains algebraic and logical developments. In Section 3, modal logic systems and rough sets are presented. Section 4 deals with membership function based MF-rough sets. Rough set models of various algebras are presented in Section 5. Section 6 contains some concluding remarks.

2. ALGEBRAIC AND LOGICAL DEVELOPMENTS

In this section, we review some abstract algebraic structures which were developed in the context of rough set theory. The Hilbert type logic systems corresponding to some of the algebras will also be presented.

In [1], the authors proposed two algebraic structures viz. pre-rough algebra and rough algebra in the framework of rough set theory specially based on the notions of rough inclusion and rough equality. It has been described in the same paper as follows. Let $\langle U, R\rangle$ be an approximation space. Two subsets P and Q of U are said to be roughly equal if $\underline{P}_R = \underline{Q}_R$ and $\overline{P}^R = \overline{Q}^R$. An equivalence relation $\approxeq$ is defined in 2^U, the power set of U, as $P \approxeq Q$ if and only if P and Q are roughly equal. Each equivalence class $[P]_{\approxeq}$ of $2^U/\approxeq$ is called a rough set (see introduction). Using these rough sets and suitable operations $\sqcup$, $\sqcap$, $\neg$ and I, $\langle 2^U/\approxeq, \sqcup, \sqcap, \neg, I, [\emptyset]_{\approxeq}, [U]_{\approxeq}\rangle$ is a pre-rough algebra, a little bit more, a rough algebra. The operations $\sqcup, \sqcap, \neg$ and I are defined as

$$[P]_{\approxeq} \sqcap [Q]_{\approxeq} = [P \sqcap Q]_{\approxeq},$$

$$[P]_{\approxeq} \sqcup [Q]_{\approxeq} = [P \sqcup Q]_{\approxeq},$$

$$\neg[P]_{\approxeq} = [\neg P]_{\approxeq},$$

$$I[P]_{\approxeq} = [IP]_{\approxeq},$$

where

$$P \sqcap Q = (P \cap Q) \cup \left(P \cap \overline{Q}^R \cap \left(\overline{P \cap Q}^R\right)^c\right),$$

$$P \sqcup Q = (P \cup Q) \cap \left(P \cup \underline{Q}_R \cup \left(\underline{P \cup Q}_R \right)^c \right),$$

$$\neg P = P^c,$$

$$IP = \underline{P}_R,$$

$\cap$, $\cup$ and c being the set theoretic intersection, union and complementation. The lattice order $\sqsubseteq$ in the above pre-rough algebra is given by $[P]_{\approx} \sqsubseteq [Q]_{\approx}$ if and only if P is roughly included in Q, i.e., $\underline{P}_R \subseteq \underline{Q}_R$ and $\overline{P}^R \subseteq \overline{Q}^R$. Thereafter, features were abstracted from $\left\langle 2^U/\approx, \sqcup, \sqcap, \neg, I, [\emptyset]_{\approx}, [U]_{\approx} \right\rangle$ to yield abstractly pre-rough algebra and rough algebra.

2.1 Algebras

To begin with, it is necessary to define quasi-Boolean algebra (**qBa**). qBa is short of Boolean algebra in that the law of excluded middle (and hence the law of contradiction) does not hold in it. In fact, in place of complementation here is taken the quasi-complementation about which more details will be discussed in Section 5. It is interesting to note that while the subsets of a set (the universal set U) form a Boolean algebra, the rough sets in U form a quasi-Boolean algebra.

Formally, it is defined by:

Definition 1. [34] *A quasi-Boolean algebra (qBa) is an abstract structure* $\langle U, \wedge, \vee, \neg, 0, 1 \rangle$ *where*

1. $\langle U, \vee, \wedge, 0, 1 \rangle$ *is a bounded distributive lattice*
2. $\neg\neg x = x$, *for all x in U*
3. $\neg(x \vee y) = \neg x \wedge \neg y$, *for all x, y in U.*

We now proceed to the main definition related with Pawlakian rough sets.

Definition 2. [1] *A* ***pre-rough algebra*** *is an abstract structure* $\langle U, \wedge, \vee, \neg, I, 0, 1 \rangle$, *where I is a unary operator on U with the following conditions:*

1. $\langle U, \wedge, \vee, \neg, 0, 1 \rangle$ *is a qBa.*
2. $I1 = 1$.
3. $I(x \wedge y) = Ix \wedge Iy$, *for all* $x, y \in U$.

4. *$Ix \leq x$, for all $x \in U$ ($\leq$ is the lattice order).*
5. *$IIx = Ix$, for all $x \in U$.*
6. *$CIx = Ix$, for all $x \in U$, where $Cx = \neg I \neg x$.*
7. *$\neg Ix \vee Ix = 1$, for all $x \in U$.*
8. *$I(x \vee y) = Ix \vee Iy$, for all $x, y \in U$.*
9. *$Cx < Cy$ and $Ix \leq Iy$ imply $x \leq y$, for all $x, y \in U$.*

Definition 3. [*1*] *Let $\langle U, \wedge, \vee, \neg, I, 0, 1\rangle$ be pre-rough algebra. Then, it is said to be a rough algebra if the sub algebra $I(U) = \{Iu : u \in U\}$ is complete and completely distributive, i.e., for any subset X of $I(U)$, lub of X and glb of X exist and for any subset $\{x_{ij} : i \in I, ,j \in J\}$ of $I(U)$,*

$$\bigwedge_{i\in I}\bigvee_{j\in J} x_{i,\ j} = \bigvee_{f:I\to J}\bigwedge_{i\in I} x_{i,\ f(i)} \text{ holds, } I, J \text{ being index sets.}$$

In the above two algebras, a binary operation $\Rightarrow$, called rough implication needs to be defined in terms of other operations satisfying the property ($P_{\Rightarrow}$):

$$x \leq y \text{ if and only if } x \Rightarrow y = 1, \text{ for all } x, y \in U.$$

The rough implication that was defined in [1] is

$$x \Rightarrow y = (\neg Ix \vee Iy) \wedge (\neg Cx \vee Cy), \text{ for all } x, y \in U.$$

It has a natural interpretation in the field of classical rough set theory. In fact, it corresponds to the notion of rough inclusion [28] viz. a subset P is roughly included in a subset Q with respect to the approximation space (U, R), P, $Q \subseteq$ U if $\underline{P}_R \subseteq \underline{Q}_R$ and $\overline{P}^R \subseteq \overline{Q}^R$. It has another importance for developing logic systems corresponding to pre-rough algebra and rough algebra. In a Hilbert type logic system corresponding to an abstract algebra, it is crucial to have an implication ($\Rightarrow$) which is interpreted in the corresponding algebra as the operation $\Rightarrow$ having the property ($P_{\Rightarrow}$). We shall discuss about the logic systems corresponding to pre-rough algebra and rough algebra in the next subsection.

In [1], a predecessor of pre-rough algebra, and of course rough algebra, has been highlighted and called topological quasi-Boolean algebra (tqBa). It is the algebra satisfying the conditions from 1 to 6 of pre-rough algebra only (Definition 2). The nomenclature of this algebra comes from topological Boolean algebra that was already known since 1944 [34]. A topological Boolean algebra is a Boolean algebra endowed with an interior operator

I satisfying the conditions from 2 to 5 of pre-rough algebra. Thus, a tqBa which is based on quasi-Boolean algebra (not necessarily a Boolean algebra) possesses one more axiom, viz. $CIx = Ix$. This axiom is obviously equivalent to $Ix \leq CIx$ and $CIx \leq Ix$ in which the second one is nothing but the algebraic counter part of modal axiom S_5 [18]. Here it may be stated that the properties 4 and 5 of pre-rough algebra are algebraic versions of modal axioms T and S_4 respectively. Also the counterpart of modal axiom B is $CIx \leq x$. In view of this, the author(s) of [45, 36] split the original notion of topological quasi-Boolean algebra, to make it more appropriate in nomenclature, into two notions viz. topological quasi-Boolean algebra and topological quasi-Boolean algebra 5 (**tqBa5**). Henceforth, in this paper, a tqBa means the abstract algebra satisfying the conditions from 1 to 5 of pre-rough algebra whereas tqBa + 6 is the abstract algebra tqBa5. In [36], it has been proved that in a tqBa5, axiom 5 of pre-rough algebra: $IIx = Ix$ is redundant. Also the algebraic counter part of modal axiom B, i.e., $CIx \leq x$ holds in a tqBa5. A natural question now arises—what would be the logics corresponding to these structures tqBa and tqBa5? Unfortunately, no affirmative response can be made on this issue with respect to the Hilbert type logic system corresponding to these algebras. This has been presented in [5]. In this project report, the author has shown that no binary operation $\Rightarrow$ can in general be defined in terms of other operations obeying the property ($P_{\Rightarrow}$) in these two algebras. The example that was constructed for the purpose is as follows.

Example 1. Let $U = \{0, x, y, 1\}$. Hasse diagram of the lattice is given in Figure 1. $\neg$ is defined as $\neg x = x$, $\neg y = y$, $\neg 1 = 0$, $\neg 0 = 1$ and I is defined as the identity operator, i.e., $Iz = z$, for all z. Then $\langle U, \wedge, \vee, \neg, I, 0, 1\rangle$ is a tqBa as well as tqBa5. Now $x \Rightarrow x$ should be an element involving x, $\neg$, $\wedge$, $\vee$ and I only. In this example $\neg x = x$, $x \wedge x = x$, $x \vee x = x$, $Ix = x$ and hence $x \Rightarrow x = x (\neq 1)$ but $x \leq x$.

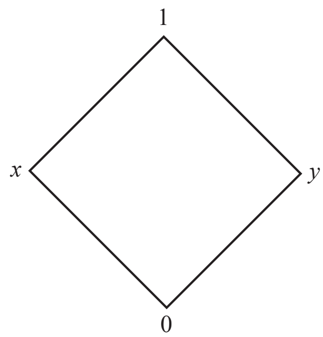

Fig. 1: Hasse diagram (tqBa, tqBa5)

In view of this example, it is clear that the three properties 7, 8 and 9 of pre-rough algebra which do not hold in tqBa5 have a crucial role for developing the Hilbert type logic system of pre-rough algebra. These properties are called intermediate property 1 (**IP1**), intermediate property 2 (**IP2**) and intermediate property 3 (**IP3**) respectively. In [45,36], an initiative was taken to check whether these axioms are independent or not in the context of pre-rough algebra. In fact, the authors of [36] proved that tqBa5 + IP1 + IP3 implies IP2. Besides this, they have shown that some axioms of pre-rough algebra like $I1 = 1$, $IIx = Ix$, $CIx = Ix$ are also deducible from other axioms. As a result, a simplified form of pre-rough algebra has been defined in [36]. Moreover, using these three intermediate properties three algebras were defined in [45,37]. They are tqBa5 + IP1 called intermediate algebra of type 1 (**IA1**), tqBa5 + IP2 called intermediate algebra of type 2 (**IA2**) and tqBa5 + IP3 called inter-mediate algebra of type 3 (**IA3**). As Example 1 becomes an instance of IA2 as well as IA3, no Hilbert type logic system corresponding to IA2 and IA3 can be developed [37]. Whether such logic system corresponding to IA1 can be constructed or not is unsolved till now.

We have already mentioned that no $\Rightarrow$ satisfying the property ($P_{\Rightarrow}$) is available in tqBa5 but such an operation (rough implication) is present in pre-rough algebra. So, a natural question: can we construct some algebraic structures in the vicinity of pre-rough algebra where rough implication would be available? On this issue, a sufficient amount of work has been done in [36]. In this paper, the authors have developed a cluster of algebras weaker than pre-rough algebra where rough implication exists. An important result that helps to construct such algebras is the following.

Proposition 1. [*36,35*] *In an algebraic structure based on qBa* (*with two unary operators* I *and* C, $C = \neg I \neg$), *the following are the necessary and sufficient conditions for the rough implication* $\Rightarrow$ *to satisfy the property* ($P_{\Rightarrow}$).

1. $\neg Ix \vee Ix = 1$
2. $x \leq y$ *implies* $Ix \leq Iy$
3. $Cx \leq Cy$ *and* $Ix \leq Iy$ *imply* $x \leq y$.

Thus, from the above Proposition 1, IP1 and IP3 are essential for obtaining rough implication. Since, $I(x \wedge y) = Ix \wedge Iy$ gives '$x \leq y$ implies $Ix \leq Iy$', the authors of [36] presented two basic structures using the properties '$I(x \wedge y) = Ix \wedge Iy$' and '$x \leq y$ implies $Ix \leq Iy$'. The steps that they took in this regard are as follows.

Definition 4. *An abstract algebra* $\langle U, \wedge, \vee, \neg, I, 0, 1\rangle$ *is said to be a* ***System0 algebra*** *if and only if*

1. $\langle U, \wedge, \vee, \neg, 0, 1\rangle$ *is a qBa.*
2. $I1 = 1$.
3. $x \leq y$ *implies* $Ix \leq Iy$.

Definition 5. *An abstract algebra* $\langle U, \wedge, \vee, \neg, I, 0, 1\rangle$ *is said to be a* ***SystemI algebra*** *if and only if* $\langle U, \wedge, \vee, \neg, I, 0, 1\rangle$ *is a System0 algebra along with IP1 and IP3.*

Definition 6. *An abstract algebra* $\langle U, \wedge, \vee, \neg, I, 0, 1\rangle$ *is said to be a* ***System*II** ***algebra*** *if and only if*

1. $\langle U, \wedge, \vee, \neg, 0, 1\rangle$ *is a qBa.*
2. $I1 = 1$.
3. $I(x \wedge y) = Ix \wedge Iy$.
4. *IP1 and IP3 hold.*

It is clear that any SystemII algebra is a SystemI algebra. But the converse, i.e., whether a SystemI algebra is a SystemII algebra or not is still open [36]. Afterwards, it has been shown in the same paper that modal axioms T: $Ix \leq x$, B: $CIx \leq x$, S_4 : $Ix \leq IIx$, S_5: $CIx \leq Ix$ do not hold in general in a SystemII algebra and hence in a SystemI algebra too. The example that was considered to show this is as follows.

Example 2. A lattice $U = \{0, x, y, u, v, 1\}$ whose Hasse diagram is shown in Figure 2 and $\neg$, I are defined in the tables given below.

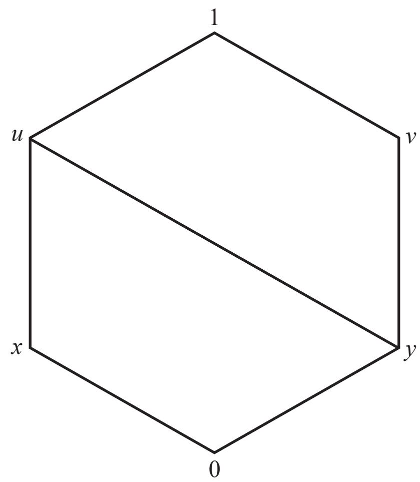

Fig. 2: Hasse diagram (SystemII algebra)

	0 x y u v 1
¬	1 v u y x 0
I	0 v x 1 x 1
C	0 v 0 v x 1

U along with the above operations is a SystemII algebra. In this example, $Iu \not\leq u, CIu \not\leq u, Iy \not\leq IIy, CIy \not\leq Iy$.

The authors further noticed that if modal axiom T is added with SystemI or SystemII algebra then it becomes a pre-rough algebra straightway. So, they added modal axioms B, S_4, S_5 separately to a SystemI and SystemII algebras. According to them, SystemIB algebra, SystemI4 algebra, SystemI5 algebra are respectively SystemI algebra + modal axiom B, SystemI algebra + S_4 and SystemI algebra + S_5. Similar is the case for the other structures SystemIIB algebra, SystemII4 algebra, SystemII5 algebra. Besides this, to obtain stronger structures they replaced ≤ by = in the modal axioms S_4, S_5 and added them to a SystemI and SystemII algebra as before. As a result, SystemI4E algebra (SystemI algebra + $IIx = Ix$) and similar other algebras SystemI5E, SystemII4E, SystemII5E are available in [36]. In the same paper relationships among the algebras were studied and presented. For a clear understanding of the various algebraic structures discussed so far, we refer to Figure 3 on page 240. Of these, no implication ⇒ can in general be defined in terms of other operations satisfying the property ($P_{\Rightarrow}$) in the bold faced algebras except for $\mathbb{IA}1$ where availability of such implication is unsolved till now. For the remaining algebras, the rough implication works smoothly.

In our paper [43], an initiative has been taken to obtain proper set theoretic rough set models for some of the above algebras prior to pre-rough algebra. The phrase 'proper set theoretic rough set model' means that it should be a set model and should not reduce to a pre-rough algebra. In fact, for any approximation space $\langle U, R \rangle$ $\langle 2^U/\approx, \sqcap, \sqcup, \neg, I, [\emptyset]_\approx, [U]_\approx \rangle$ becomes a pre-rough algebra and hence it is not a proper set theoretic rough set model of any algebra weaker than pre-rough algebra. For proper set theoretic rough set models, it is necessary to check which properties of I are available in the aforesaid algebras. For example, in tqBa modal axioms T, $S4$ and hence axiom D ($Ix \leq Cx$) are available, whereas in tqBa5, IA1, IA2 and IA3 modal axioms T, S_4, S_5 and hence axioms D, B hold. But (in view of standard modal systems), no information is available regarding the algebraic counterpart of the modal axiom K. We have further noticed that there are two types of algebras, one in

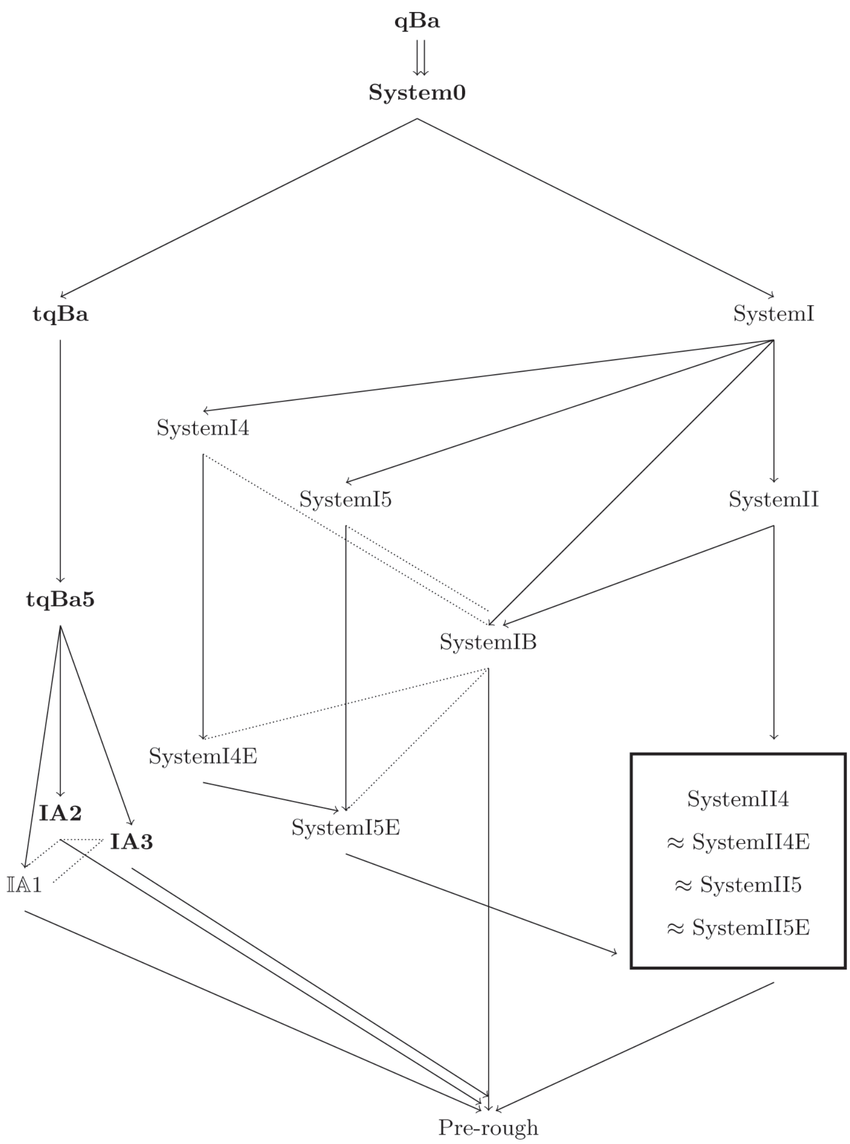

Fig. 3: Algebras in the vicinity of pre-rough algebra, P $\rightrightarrows$ Q stands for the algebra Q has one more operator and some axioms for the new operator than the algebra P. P → Q stands for both the algebras P and Q have the same operations and the algebra Q is always the algebra P. P ... Q stands for the algebras P and Q are independent.

which no implication can be defined in terms of other operations satisfying the property ($P_{\Rightarrow}$) (e.g., tqBa, tqBa5, IA2, IA3 etc.), other in which an implication (the rough implication) is available obeying the property ($P_{\Rightarrow}$) (e.g., SystemI, SystemII, SystemI4, SystemI5 etc.). As modal axiom K in the form $I(x \Rightarrow y) \Rightarrow (Ix \Rightarrow Iy) = 1$ is irrelevant for the algebras tqBa, tqBa5, IA2, IA3 etc. (as no $\Rightarrow$ is available), we consider the other form of modal axiom K: $I(\neg x \vee y) \leq \neg Ix \vee Iy$. The above form is similar to $I(A^C \cup B) \subseteq (IA)^C \cup IB$, the algebraic counterpart of the modal axiom K in Boolean base. Thereafter, we have checked whether this form of modal axiom K holds or not in the above algebras. We have shown that this axiom holds in pre-rough algebra, IA1, IA2 but does not hold in tqBa, tqBa5, IA3, System0, SystemI, SystemII etc. Later, a number of new abstract algebras based on qBa have been introduced in order to fulfil the following purposes:

- In these algebras, properties of I are enhanced in hierarchical order starting form modal axiom D (axiom K: $I(\neg x \vee y) \leq \neg Ix \vee Iy$ is not considered as it does not hold generally in our constructed rough set models) [see Section 5].
- Proper set theoretic rough set models may be constructed for these algebras.

The newly created algebras are thus:

Definition 7. *An abstract algebra* $\langle U, \wedge, \vee, \neg, I, 0, 1\rangle$, *where I is a unary operator on* U, *is said to be a semi topological quasi-Boolean algebra* (***stqBa***) *if and only if*

1. $\langle U, \wedge, \vee, \neg, 0, 1\rangle$ *is a qBa.*
2. $I1 = 1$.
3. $I(x \wedge y) = Ix \wedge Iy$, *for all* $x, y \in U$.

In this algebra modal axiom K, D and T do not hold [43].

Definition 8. *Let* $\langle U, \wedge, \vee, \neg, I, 0, 1\rangle$ *be a stqBa. Then it is said to be a semi topological quasi-Boolean algebra with modal axiom D* (***stqBaD***) *if and only if* $Ix < Cx$, *for all* $x \in U$ ($Cx = \neg I \neg x$).

The modal axiom T generally does not hold in a stqBaD [43].

Definition 9. *Let* $\langle U, \wedge, \vee, \neg, I, 0, 1\rangle$ *be a stqBa. Then it is said to be a semi topological quasi-Boolean algebra with modal axiom T* (***stqBaT***) *if and only if* $Ix \leq x$, *for all* $x \in U$.

It is obvious that a stqBaT is a stqBaD but the converse is not true. Further it has been shown that the modal axioms $B(CIx \leq x)$ and $S_4(Ix \leq IIx)$ generally do not hold in a stqBaT [43].

Definition 10. *Let* $\langle U, \wedge, \vee, \neg, I, 0, 1\rangle$ *be a stqBaT. Then it is said to be a semi topological quasi-Boolean algebra with modal axiom B* (***stqBaB***) *if and only if* $CIx \leq x$, *for all* $x \in U$ $(Cx = \neg I \neg x)$.

The modal axioms S_4 and $S_5(CIx \leq Ix)$ generally do not hold in a stqBaB [43].

A tqBa is nothing but a stqBaT + modal axiom S_4. In a tqBa the modal axioms B and S_5 do not hold [43]. The algebras stqBaB and tqBa are independent to each other.

A tqBa5 is a stqBaT + modal axiom S_5.

Figure 4 shows a relationship between the old and new algebras.

Logics and proper set theoretic rough set models of newly created algebras have been discussed in Subsection 2.2 and Section 5 respectively.

Another direction of work has been done in [37]. It has been mentioned earlier that it is not possible to define $\Rightarrow$ in terms of other operations satisfying the property $(P_{\Rightarrow})$ in a qBa (even in a tqBa5). But, to develop the Hilbert type axiomatic system corresponding to these algebras such an implication is needed. In this paper [37], such an implication operation has been imposed in qBa and some other stronger structures where this operation is not available in general. This is, in a way, similar to Rasiowa's approach in [34] where the algebraic structure called relatively pseudo-complemented lattice (now called residuated lattice) had been introduced by putting together positive implication algebra and a lattice structure. In the present case, implicative algebra and quasi-Boolean algebra have been amalgamated. Following Rasiowa [34] these structures have been named implicative quasi-Boolean algebra(IqBa) and implicative quasi-Boolean algebra with operator(**IqBaO).** The operators they [37] have taken are topological operators corresponding to the modal axioms T, S_4 and S_5 [18]. The corresponding algebras have been named as implicative quasi-Boolean algebra with modal axiom **T**(**IqBaT**), implicative quasi-Boolean algebra with modal axiom S_4(**IqBa4**) and implicative quasi-Boolean algebra with modal axiom S_5(**IqBa5**). The definitions and important features of these algebras are as follows (see [37] for details).

Definition 11. *An abstract algebra* $\langle U, \wedge, \vee, \Rightarrow, \neg, 0, 1\rangle$ *is called an implicative quasi-Boolean algebra*(*IqBa*) *if and only if*

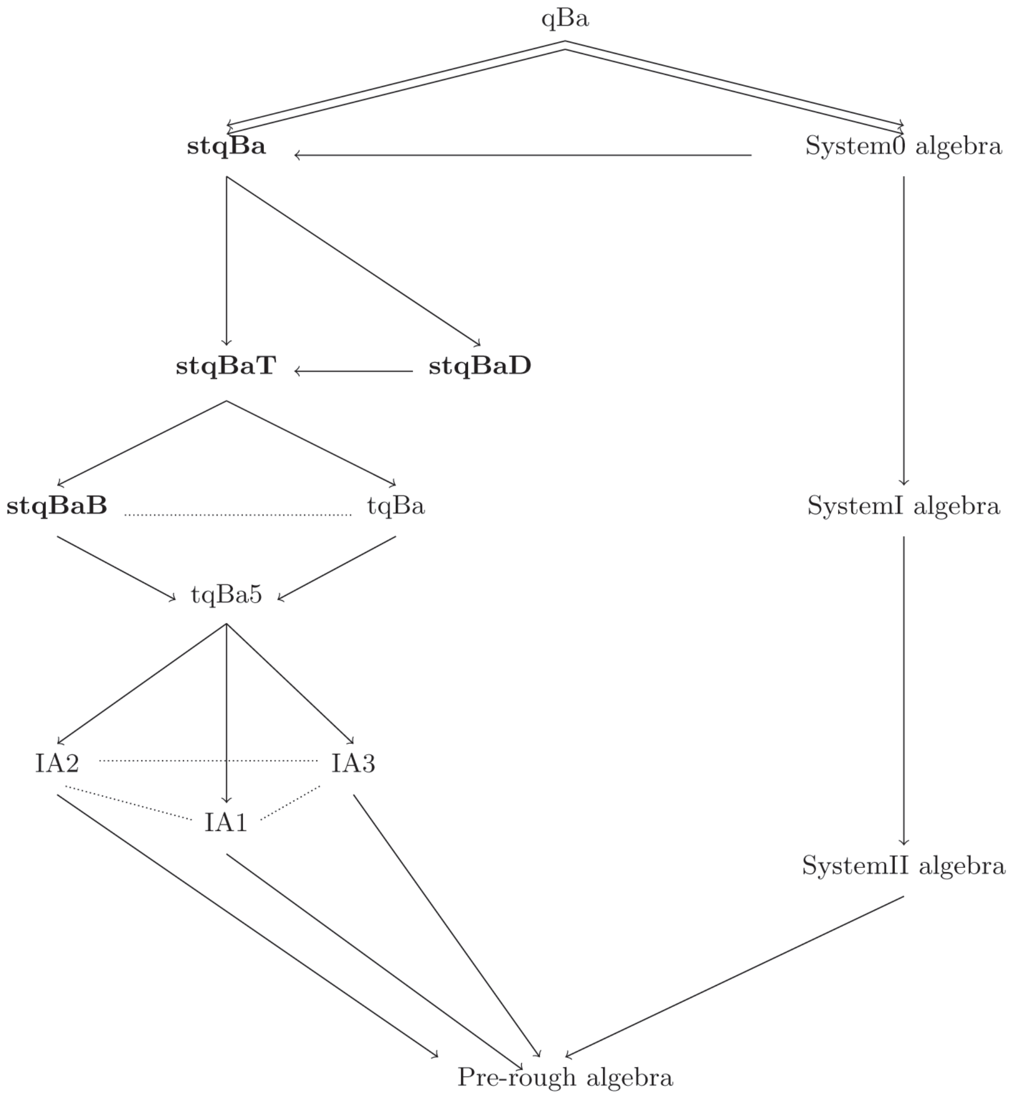

Fig. 4: Relationship diagram of the newly created algebras and old algebras. Bold faced algebras are newly introduced in our paper [43] whereas others are available in different literature. P $\rightrightarrows$ Q stands for the algebra Q contains one new operator and some axioms for the new operator than the algebra P. P → Q stands for both the algebras P and Q have the same operations but Q contains some more axioms than P. P ... Q stands for the algebras P and Q are independent to each other.

1. $\langle U, \wedge, \vee, \neg, 0, 1\rangle$ *is a qBa.*
2. $x \Rightarrow y = 1$ *if and only if* $x \leq y$, *for all* $x, y \in U$. $(P_{\Rightarrow})$

Definition 12. *An algebra* $\langle U, \wedge, \vee, \Rightarrow, \neg, I, 0, 1\rangle$, *where I is a unary operator, will be called an implicative quasi-Boolean algebra with operator* (*IqBaO*) *if and only if*

1. $\langle U, \wedge, \vee, \Rightarrow, \neg, 0, 1\rangle$ *is a IqBa.*
2. $I1 = 1$.
3. $I(x \wedge y) = Ix \wedge Iy$, *for all* $x, y \in U$.

Definition 13. *Let* $\langle U, \wedge, \vee, \Rightarrow, \neg, I, 0, 1\rangle$ *be a IqBaO. Then it will be an*

1. *implicative quasi-Boolean algebra with modal axiom T* (*IqBaT*) *if and only if* $Ix \leq x$ *holds, for all* $x \in U$ (***modal axiom T***),
2. *implicative quasi-Boolean algebra with modal axiom* S_4 (*IqBa*4) *if and only if it is a IqBaT and* $Ix \leq IIx$, *for all* $x \in U$ (***modal axiom*** S_4),
3. *implicative quasi-Boolean algebra with modal axiom* S_5 (*IqBa5*) *if and only if it is a IqBa4 and* $CIx \leq Ix$, *for all* $x \in U$, *where* $C = \neg I \neg$ (***modal axiom*** S_5).

By several examples it has been shown [37] that modal axiom K in the form $I(x \Rightarrow y) \Rightarrow (Ix \Rightarrow Iy)$ does not hold generally in these algebras. As earlier, it has also been mentioned [37] that the axiom $Ix \leq IIx$ is redundant in a IqBa5 and the modal axiom $B(CIx \leq x)$ also follows in this algebra. The authors of [37] also observed the followings.

- A IqBa5 is a tqBa5 algebra along with an implication having the property $(P_{\Rightarrow})$: $x \Rightarrow y = 1$ if and only if $x \leq y$ for all x, y.
- If the above implication is defined by $x \Rightarrow_B y = \neg x \vee y$ in a qBa and the property $(P_{\Rightarrow})$ is assumed for $\Rightarrow_B$ then the qBa becomes a Boolean algebra. Hence, a IqBa5 then turns into a topological Boolean algebra [34] (also known as an interior algebra [7] with S_5 axiom).
- If the above implication is defined by $x \Rightarrow_R y = (\neg Ix \vee Iy) \wedge (\neg Cx \vee Cy)$ in a IqBa5 and the property $(P_{\Rightarrow})$ is assumed for $\Rightarrow_R$ then the IqBa5 becomes a pre-rough algebra. Thus, a tqBa5 with $\Rightarrow_R$ satisfying $(P_{\Rightarrow})$ turns into a pre-rough algebra.
- If the above implication is defined by $x \Rightarrow_L y = (C\neg x \vee y) \wedge (\neg x \vee Cy)$ [8] in a IqBa5 and the property $(P_{\Rightarrow})$ is assumed for $\Rightarrow_L$ then the IqBa5 becomes a 3-valued

Lukasiewicz (Moisil) algebra [8]. Thus, a tqBa5 with $\Rightarrow_L$ satisfying ($P_{\Rightarrow}$) turns into a 3-valued Lukasiewicz (Moisil) algebra.

Subsequently, we have expanded the area considering the three intermediate properties IP1, IP2 and IP3. It is to be noted that the properties are separately used to define the three intermediate algebras IA1, IA2, IA3 and no implication can in general be defined in terms of other operations satisfying the property ($P_{\Rightarrow}$) in IA2, IA3 and in case of IA1, it is unsolved. We added the three intermediate properties IP1, IP2 and IP3 separately to IqBaO, IqBaT, IqBa4 and IqBa5 and investigated the consequences [44]. As a result, twelve additional algebraic structures had been obtained as shown in Figure 5. Of these, the chain of algebras qBa, IqBa, IqBaO, IqBaT, IqBa4 and IqBa5 (bold face) are included in [37]. In fact, we have actually added the modal axiom T to IqBa1 to obtain IqBa1,T which is the same as adding IP1 to IqBaT. Similar is the case for all other structures.

We now present a brief discussion about the algebras just mentioned above (see [44] for details).

Definition 14. *Let* $\langle U, \wedge, \vee, \Rightarrow, \neg, I, 0, 1\rangle$, *be a IqBaO. Then it is said to be an*

1. *implicative quasi-Boolean algebra with IP1* (*IqBa1*) *if and only if* $\neg Ix \vee Ix = 1$ *holds, for all* $x \in U$,
2. *implicative quasi-Boolean algebra with IP2* (*IqBa2*) *if and only if* $I(x \vee y) = Ix \vee Iy$ *holds, for all* $x, y \in U$,
3. *implicative quasi-Boolean algebra with IP3* (*IqBa3*) *if and only if* $Cx \leq Cy$ *and* $Ix \leq Iy$ *imply* $x \leq y$, *for all* $x, y \in U$.

By several examples, independence of the algebras IqBa1, IqBaT, IqBa2 and IqBa3 has been established.

Definition 15. *Let* $\langle U, \wedge, \vee, \Rightarrow, \neg, I, 0, 1\rangle$, *be a IqBaO. Then it is said to be an*

1. *implicative quasi-Boolean algebra with IP1 and modal axiom T* (*IqBa1,T*) *if and only it is a IqBa1 and* $Ix \leq x$, *for all x E U*,
2. *implicative quasi-Boolean algebra with IP2 and modal axiom T* (*IqBa2,T*) *if and only if it is a IqBa2 and* $Ix \leq x$, *for all* $x \in U$,
3. *implicative quasi-Boolean algebra with IP3 and modal axiom T* (*IqBa3,T*) *if and only if it is a IqBa3 and* $Ix \leq x$, *for all* $x \in U$.

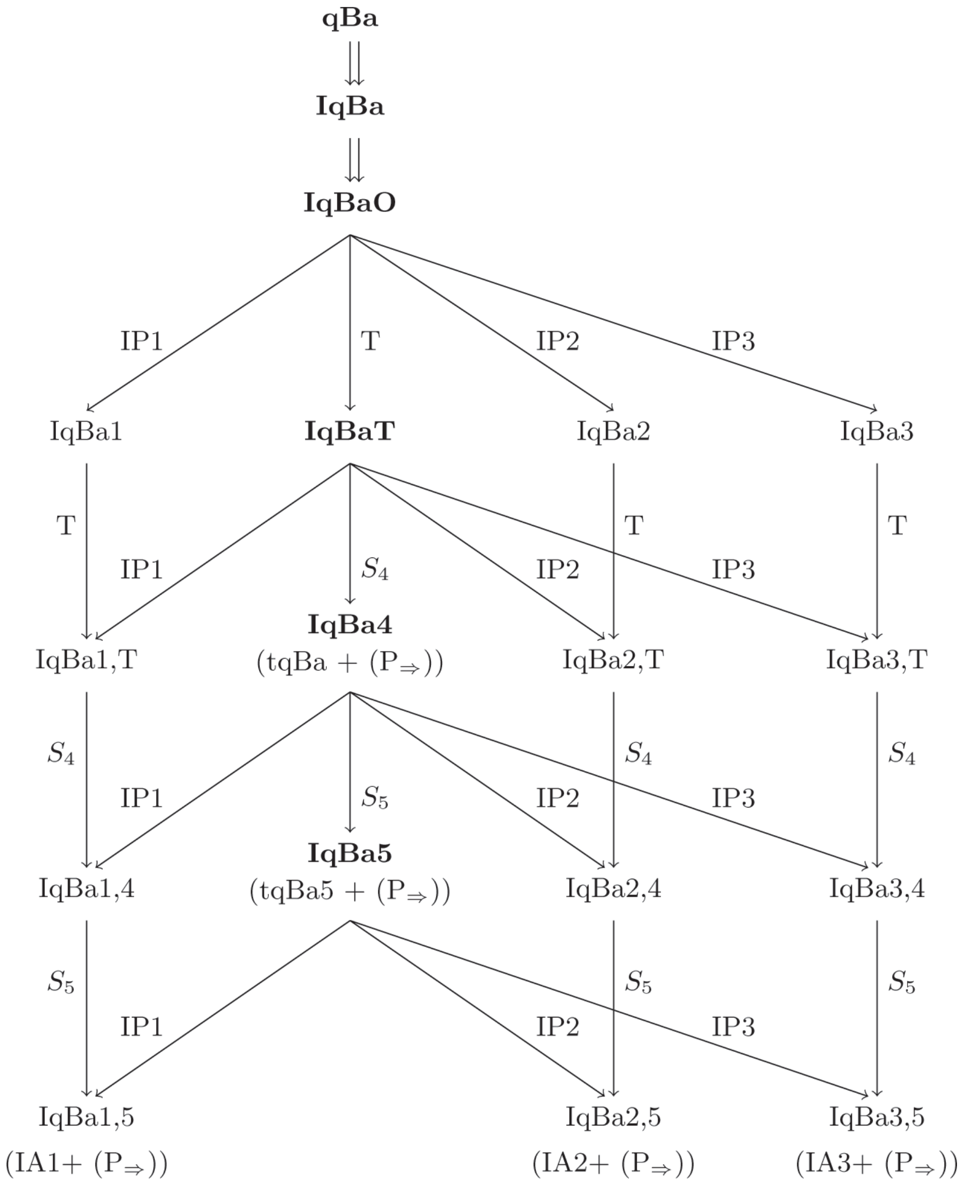

Fig. 5: Algebras based on IqBaO

P $\rightrightarrows$ Q stands for the algebra Q has one more operation than the algebra P. P → Q stands for both the algebras P and Q have the same operations but Q has one more axiom than P.

As before, independence of the algebras IqBa1,T, IqBa4, IqBa2,T and IqBa3,T has also been shown.

Definition 16. *Let* $\langle U, \wedge, \vee, \Rightarrow, \neg, I, 0, 1 \rangle$ *be a IqBaO. Then it is said to be an*

1. *implicative quasi-Boolean algebra with IP1 and modal axiom S4* (*IqBa1*,*4*) *if and only it is a IqBa1*,*T and* $Ix \leq IIx$, *for all* $x \in U$,
2. *implicative quasi-Boolean algebra with IP2 and modal axiom S4* (*IqBa2*,*4*) *if and only if it is a IqBa2*,*T and* $Ix \leq IIx$, *for all* $x \in U$,
3. *implicative quasi-Boolean algebra with IP3 and modal axiom S4* (*IqBa3*,*4*) *if and only if it is a IqBa3*,*T and* $Ix \leq IIx$, *for all* $x \in U$.

Independence issue of the algebras IqBa1,4, IqBa2,4, IqBa3,4 along with IqBa5 has been established in the same paper [44] with the help of some examples.

Definition 17. *Let* $\langle U, \wedge, \vee, \Rightarrow, \neg, I, 0, 1 \rangle$ *be a IqBaO. Then it is said to be an*

1. *implicative quasi-Boolean algebra with IP1 and modal axiom S5* (*IqBa1*,*5*) *if and only if it is a IqBa1*,*4 and* $CIx \leq Ix$, *for all* $x \in U$,
2. *implicative quasi-Boolean algebra with IP2 and modal axiom S5* (*IqBa2*,*5*) *if and only if it is a IqBa2*,*4 and* $CIx \leq Ix$, *for all* $x \in U$,
3. *implicative quasi-Boolean algebra with IP3 and modal axiom S5* (*IqBa3*,*5*) *if and only if it is a IqBa3*,*4 and* $CIx \leq Ix$, *for all* $x \in U$.

That IqBa1,5, IqBa2,5 and IqBa3,5 are independent algebras is shown in [44].

It is to be noted that if implication were imposed (satisfying the property ($P_{\Rightarrow}$) directly in IA1, IA2 and IA3 then IA1 + ($P_{\Rightarrow}$), IA2 + ($P_{\Rightarrow}$) and IA3 + ($P_{\Rightarrow}$) would be the same with the algebras IqBa1,5, IqBa2,5 and IqBa3,5 respectively.

2.2 Logics

In this section logics corresponding to the algebras discussed in subsection 2.1 will be considered. We present mainly the Hilbert type logic system. The Sequent Calculi for most of the algebras are available in various literature [46, 45, 36, 37, 43].

The Hilbert System for pre-rough algebra: In [1], the formal system of pre-rough algebra has already been developed. However, the number of axioms of pre-rough algebra has been reduced in [37]. As a consequence, the number of axioms of pre-rough logic has also been reduced in the same paper. The logic $\mathcal{L}_{PRA}$ of the modified pre-rough algebra is as follows [37]. The alphabet of the language of $\mathcal{L}_{PRA}$ consists of

— propositional variables $p, q, r, \ldots$

— unary logical connectives $\neg$ and I.

— binary logical connective $\wedge$.

— parentheses (,).

Well formed formulas (wffs) are formed in the usual way and $\alpha, \beta, \gamma, \delta$ etc. are used to denote them.

$\vee$ (binary), $\Rightarrow$ (binary) and C (unary) are definable logical connectives:

$\alpha \vee \beta \equiv \neg(\neg\alpha \wedge \beta)$, $\alpha \Rightarrow \beta \equiv (\neg I\alpha \vee I\beta) \wedge (C\alpha \vee C\beta)$, $C\alpha \equiv \neg I\neg\alpha$, for any wffs α, β of $\mathcal{L}_{PRA}$.

Axioms for $\mathcal{L}_{PRA}$:

1. $\alpha \Rightarrow \neg\neg\alpha$
2. $\neg\neg\alpha \Rightarrow \alpha$
3. $\alpha \wedge \beta \Rightarrow \beta$
4. $\alpha \wedge \beta \Rightarrow \beta \wedge \alpha$
5. $\alpha \wedge (\beta \vee \gamma) \Rightarrow (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$
6. $(\alpha \wedge \beta) \vee (\alpha \wedge \gamma) \Rightarrow \alpha \wedge (\beta \vee \gamma)$
7. $I\alpha \Rightarrow \alpha$
8. $I\alpha \wedge I\beta \Rightarrow I(\alpha \wedge \beta)$

Rules of inference:

1. $\dfrac{\alpha,\ \alpha \Rightarrow \beta}{\beta}$ Modus ponens (MP)

2. $\dfrac{\alpha \Rightarrow \beta,\ \beta \Rightarrow \gamma}{\alpha \Rightarrow \gamma}$ Hypothetical syllogism (HS)

3. $\dfrac{\alpha}{\beta \Rightarrow \alpha}$

4. $\dfrac{\alpha \Rightarrow \beta}{\neg\beta \Rightarrow \neg\alpha}$

5. $\dfrac{\alpha \Rightarrow \beta,\ \alpha \Rightarrow \gamma}{\alpha \Rightarrow \beta \wedge \gamma}$

6. $\dfrac{\alpha \Rightarrow \beta,\ \beta \Rightarrow \alpha,\ \gamma \Rightarrow \delta,\ \delta \Rightarrow \gamma}{(\alpha \Rightarrow \gamma) \Rightarrow (\beta \Rightarrow \delta)}$

7. $\dfrac{\alpha \Rightarrow \beta}{I\alpha \Rightarrow I\beta}$

8. $\dfrac{\alpha}{I\alpha}$ Necessitation (N)

9. $\dfrac{I\alpha \Rightarrow I\beta,\ C\alpha \Rightarrow C\beta}{\alpha \Rightarrow \beta}$

$\vdash \alpha$ stands for α is a theorem in the logic system $\mathcal{L}_{PRA}$ as usual sense.

Definition 18. *A model of* $\mathcal{L}_{PRA}$ *is* (U, v) *where* U = $\langle U,\ \wedge,\ \vee,\ \Rightarrow,\ \neg,\ I,\ 0,\ 1\rangle$ is a *pre-rough algebra and v is a valuation function which assigns a value* $v(p) \in U$ *for each atomic wff p of* $\mathcal{L}_{PRA}$.

Remark 1. Any valuation function v can be extended to arbitrary formulae as follows

$$(\alpha \wedge \beta) = v(\alpha) \wedge v(\beta),\ v(\neg\alpha) = \neg v(\alpha), v(\alpha \Rightarrow \beta) = v(\alpha) \Rightarrow v(\beta), v(I\alpha) = Iv(\alpha).$$

As ∨ and C are definable connectives, it can be shown that $v(\alpha \vee \beta) = v(\alpha) \vee v(\beta), v(C\alpha) = Cv(\alpha)$ where $Cx = \neg I \neg x$.

Definition 19. *A wff α is said to be true in a model* $\langle \mathrm{U}, v\rangle$ *of* $\mathcal{L}_{PRA}$ *if and only if* $v(\alpha) = 1$.

Definition 20. *A wff α is said to be valid in the class of all models of* $\mathcal{L}_{PRA}$ *if and only if α is true in every model* $\langle \mathrm{U}, v\rangle$ *of* $\mathcal{L}_{PRA}$.

Remark 2. A wff $\alpha \Rightarrow \beta$ is valid if and only if $v(\alpha) \leq v(\beta)$, for all models $\langle \mathrm{U}, v\rangle$ of $\mathcal{L}_{PRA}$.

Theorem 1. (*Soundness*)[*1*]*: If* $\vdash \alpha$ *in the logic system* $\mathcal{L}_{PRA}$ *then* α *is valid in the class of all models of* $\mathcal{L}_{PRA}$.

Theorem 2. (*Completeness*)[*1*]*: If* α *is valid in the class of all models of* $\mathcal{L}_{PRA}$ *then* $\vdash \alpha$ *in the logic system* $\mathcal{L}_{PRA}$.

The Hilbert System for rough algebra: Rough logic $\mathcal{L}_{RA}$ for rough algebra has been presented in [1].

The alphabet of the language of $\mathcal{L}_{RA}$ is the alphabet of the language of $\mathcal{L}_{PRA}$ + logical symbol $\bigvee$, standing for infinite disjunction. One definable logical symbol $\bigwedge$ (infinite conjuction) stands for $\neg \bigvee \neg$.

Formulae formation rule with respect to $\bigvee$: For any index set J, $\bigvee_{j\in J} \alpha_j$ is a wff in $\mathcal{L}_{RA}$ if and only if α_j is of the form $I\beta_j$, for some β_j, $j \in J$.

Axioms: All axioms of $\mathcal{L}_{PRA}$ along with

1. $I\alpha_j \Rightarrow \bigvee_{j\in J} I\alpha_j$, for each α_j, $j \in \mathrm{J}$,
2. $\bigvee_{j\in J} \alpha_j \Rightarrow I \bigvee_{j\in J} \alpha_j$,
3. $I \bigvee_{j\in J} \alpha_j \Rightarrow \bigvee_{j\in J} \alpha_j$,
4. $\bigvee_{j\in J} \bigwedge_{k\in K} I\alpha_{j,\,k} \Rightarrow \bigwedge_{f\in K^J} \bigvee_{j\in J} I\alpha_{j,\,f(j)}$,
5. $\bigwedge_{f\in K^J} \bigvee_{j\in J} I\alpha_{j,\,f(j)} \Rightarrow \bigvee_{j\in J} \bigwedge_{k\in K} I\alpha_{j,\,k}$,

where J, K are index sets and K^J is the set of maps of J into K.

Rules of inference: All rules of inference of $\mathcal{L}_{PRA}$ + one new rule: $\dfrac{I\alpha_i \Rightarrow I\beta}{\bigvee_{j\in J} I\alpha_j \Rightarrow I\beta}$, for each $j \in J$.

Theorem 3. [*1*] $\mathcal{L}_{RA}$ *is sound and complete relative to the class of all models of* $\mathcal{L}_{RA}$.

Remark 3.

1. No Hilbert type logic system can be constructed for the algebras tqBa, tqBa5, IA1, IA2, IA3. This is due to unavailability of $\Rightarrow$ in these algebras.
2. As Example 1 becomes an instance of the algebras stqBa, stqBaT, stqBaD and stqBaB, the Hilbert type system for the said algebras can not be developed.

Due to availability of rough arrow in SystemI algebra, SystemIB algebra, SystemI4 algebra, SystemI4E algebra, SystemI5 algebra, SystemI5E algebra SystemII algebra and SystemII4 algebra, the Hilbert type logic systems corresponding to these algebras have been developed in [36].

Let $\mathcal{L}_I$, $\mathcal{L}_{IB}$, $\mathcal{L}_{I4}$, $\mathcal{L}_{I4E}$, $\mathcal{L}_{I5}$, $\mathcal{L}_{I5E}$, $\mathcal{L}_{II}$, $\mathcal{L}_{II4}$ be the logic systems for SystemI algebra, SystemIB algebra, SystemI4 algebra, SystemI4E algebra, SystemI5 algebra, SystemI5E algebra, SystemII algebra and SystemII4 algebra respectively.

The Hilbert type System $\mathcal{L}_I$ for SystemI algebra: The language of $\mathcal{L}_I$ is the same as that of $\mathcal{L}_{PRA}$. The first six axioms and all rules of $\mathcal{L}_{PRA}$ are the axioms and rules of this system.

The Hilbert Systems $\mathcal{L}_{IB}$, $\mathcal{L}_{I4}$, $\mathcal{L}_{I4E}$, $\mathcal{L}_{I5}$, $\mathcal{L}_{I5E}$, $\mathcal{L}_{II}$, $\mathcal{L}_{II4}$: The languages of the systems $\mathcal{L}_{IB}$, $\mathcal{L}_{I4}$, $\mathcal{L}_{I4E}$, $\mathcal{L}_{I5}$, $\mathcal{L}_{I5E}$, $\mathcal{L}_{II}$, $\mathcal{L}_{II4}$ are the same as that of $\mathcal{L}_I$. In all cases, all axioms and rules of $\mathcal{L}_I$ are there together with some extra axiom(s) viz.,

$CI\alpha \Rightarrow \alpha$ for $\mathcal{L}_{IB}$,

$I\alpha \Rightarrow II\alpha$ for $\mathcal{L}_{I4}$,

$I\alpha \Rightarrow II\alpha$ and $II\alpha \Rightarrow I\alpha$ for $\mathcal{L}_{I4E}$

$CI\alpha \Rightarrow I\alpha$ for $\mathcal{L}_{I5}$,

$CI\alpha \Rightarrow I\alpha$ and $I\alpha$ for $\mathcal{L}_{I5}$,

$I(\alpha \wedge \beta) \Rightarrow I\alpha \wedge I\beta$ for $\mathcal{L}_{I1}$,

$I(\alpha \wedge \beta) \Rightarrow I\alpha \wedge I\beta$ and $I\alpha \Rightarrow II\alpha$ for $\mathcal{L}_{I14}$.

Theorem 4. [36] All *these systems $\mathcal{L}_I$, $\mathcal{L}_{IB}$, $\mathcal{L}_{I4}$, $\mathcal{L}_{I4E}$, $\mathcal{L}_{I5}$, $\mathcal{L}_{I5E}$, $\mathcal{L}_{II}$, $\mathcal{L}_{II4}$ are sound and complete relative to the class of all corresponding models.*

In each of the implicative algebras, implication has been imposed there. So, the Hilbert systems corresponding to these algebras have been constructed and are available in [37,44].

In [37], the Hilbert Systems L_h, L_O, L_T, L_4, L_5 corresponding to the algebras IqBa, IqBaO, IqBaT, IqBa4 and IqBa5 have been presented.

The Hilbert system L_h: The alphabet of the language of L_h consists of

— propositional variables p, q, r, ...

— unary logical connective $\neg$.

— binary logical connectives $\wedge$ and $\Rightarrow$.

— parentheses (,).

Well formed formulas (wffs) are formed in the usual way and denoted by α, β, γ, δ etc.

Axioms for L_h:

1. $\alpha \Rightarrow \neg\neg\alpha$
2. $\neg\neg\alpha \Rightarrow \alpha$
3. $\alpha \wedge \beta \Rightarrow \beta$
4. $\alpha \wedge \beta \Rightarrow \beta \wedge \alpha$
5. $\alpha \wedge (\beta \vee \gamma) \Rightarrow (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$
6. $(\alpha \wedge \beta) \vee (\alpha \wedge \gamma) \Rightarrow \alpha \wedge (\beta \vee \gamma)$

Rules of inference:

1. $\dfrac{\alpha,\ \alpha \Rightarrow \beta}{\beta}$ Modus ponens (MP)

2. $\dfrac{\alpha \Rightarrow \beta,\ \beta \Rightarrow \gamma}{\alpha \Rightarrow \gamma}$ Hypothetical syllogism (HS)

3. $\dfrac{\alpha}{\beta \Rightarrow \alpha}$

4. $\dfrac{\alpha \Rightarrow \beta}{\neg\beta \Rightarrow \neg\alpha}$

5. $\dfrac{\alpha \Rightarrow \beta,\ \alpha \Rightarrow \gamma}{\alpha \Rightarrow \beta \wedge \gamma}$

6. $\dfrac{\alpha \Rightarrow \beta,\ \beta \Rightarrow \alpha,\ \lambda \Rightarrow \delta,\ \delta \Rightarrow \gamma}{(a \Rightarrow \gamma) \Rightarrow (\beta \Rightarrow \delta)}$

The Hilbert systems L_O, L_T, L_4, L_5: The alphabets of the language of L_O, L_T, L_4, L_5 are the same and that is: the alphabet of the language of L_h with one additional logical connective I. C is a definable connective where $C = \neg I \neg$.

$I\alpha$ is a wff if α is so.

Axioms for L_O, L_T, L_4, L_5:

All axioms of $L_h + I\alpha \wedge I\beta \Rightarrow I(\alpha \wedge \beta)$ for L_O.

All axioms of $L_h + I\alpha \wedge I\beta \Rightarrow I(\alpha \wedge \beta) + I\alpha \Rightarrow \alpha$ for L_T.

All axioms of $L_h + I\alpha \wedge I\beta \Rightarrow I(\alpha \wedge \beta) + I\alpha \Rightarrow \alpha + I\alpha \Rightarrow II\alpha$ for L_4.

All axioms of $L_h + I\alpha \wedge I\beta \Rightarrow I(\alpha \wedge \beta) + I\alpha \Rightarrow \alpha + CI\alpha \Rightarrow I\alpha$ for L_5.

Rules for L_O, L_T, L_4, L_5: In all cases, rules are the same and that is: the Rules of inference of L_h along with

$$\frac{\alpha \Rightarrow \beta}{I\alpha \Rightarrow I\beta} \text{ and } \frac{\alpha}{I\alpha} \text{ Necessitation } (\mathrm{N}).$$

In [44], L_1, L_2, L_3, $L_{1,T}$, $L_{2,T}$, $L_{3,T}$, $L_{1,4}$, $L_{2,4}$, $L_{3,4}$, $L_{1,5}$, $L_{2,5}$, $L_{3,5}$ are the logic systems for the algebras IqBa1, IqBa2, IqBa3, IqBa1,T, IqBa2,T, IqBa3,T, IqBa1,4, IqBa2,4, IqBa3,4, IqBa1,5, IqBa2,5 and IqBa3,5 respectively.

Hilbert systems L_1, $L_{1,T}$, $L_{1,4}$, $L_{1,5}$: The alphabets of the language of L_1, $L_{1,T}$, $L_{1,4}$, $L_{1,5}$ are the same with the alphabet of the language of L_O.

Axioms for L_1, $L_{1,T}$, $L_{1,4}$, $L_{1,5}$:

All axioms of $L_O + \neg I\alpha \vee I\alpha$ for L_1.

All axioms of $L_1 + I\alpha \Rightarrow \alpha$ for $L_{1,T}$.

All axioms of $L_{1,T} + I\alpha \Rightarrow II\alpha$ for $L_{1,4}$.

All axioms of $L_{1,T} + CI\alpha \Rightarrow I\alpha$ for $L_{1,5}$.

Rules for L_1, $L_{1,T}$, $L_{1,4}$, $L_{1,5}$: In all cases, rules are the same with the rules of L_O.

The Hilbert systems L_2, $L_{2,T}$, $L_{2,4}$, $L_{2,5}$: The alphabets of the language of L_2, $L_{1,T}$, $L_{2,4}$, $L_{2,5}$ are the same with the alphabet of the language of L_O.

Axioms for L_2, $L_{1,T}$, $L_{2,4}$, $L_{2,5}$:

All axioms of L_O + $I(\alpha \vee \beta) \Rightarrow I\alpha \vee I\beta$ for L_2.

All axioms of L_2 + $I\alpha \Rightarrow \alpha$ for $L_{2,T}$.

All axioms of $L_{2,T}$ + $I\alpha \Rightarrow II\alpha$ for $L_{2,4}$.

All axioms of $L_{2,T}$ + $CI\alpha \Rightarrow I\alpha$ for $L_{2,5}$.

Rules for L_2, $L_{2,T}$, $L_{2,4}$, $L_{2,5}$: In all cases, rules are the same with the rules of L_O.

Hilbert systems L_3, $L_{3,T}$, $L_{3,4}$, $L_{3,5}$: The alphabets of the language of L_3, $L_{3,T}$, $L_{3,4}$, $L_{3,5}$ are the same with the alphabet of the language of L_O.

Axioms for L_3, $L_{3,T}$, $L_{3,4}$, $L_{3,5}$:

All axioms of L_O are the axioms of L_3.

All axioms of L_3 + $I\alpha \Rightarrow \alpha$ for $L_{3,T}$.

All axioms of $L_{3,T}$ + $I\alpha \Rightarrow II\alpha$ for $L_{3,4}$.

All axioms of $L_{3,T}$ + $CI\alpha \Rightarrow I\alpha$ for $L_{3,5}$.

Rules for L_3, $L_{3,T}$, $L_{3,4}$, $L_{3,5}$: In all cases, rules are the same with the rules of L_O along with one new rule:

$$\frac{I\alpha \Rightarrow I\beta,\ C\alpha \Rightarrow C\beta}{\alpha \Rightarrow \beta}.$$

Theorem 5. [*37*, *44*] *With respect to the class of corresponding models the above Hilbert Systems* L, L_O, L_T, L_4, L_5, L_1, $L_{1,T}$, $L_{1,4}$, $L_{1,5}$, L_2, $L_{2,T}$, $L_{2,4}$, $L_{2,5}$, L_3, $L_{3,T}$, $L_{3,4}$, $L_{3,5}$ *are sound and complete.*

3. MODAL LOGIC SYSTEMS AND ROUGH SETS

The standard normal modal systems are K, D, T, S_4, B, S_5. These are classical propositional logics enhanced by modal operators L (necessity) and M (possibility). Along with the axioms of propositional logic, modal axioms are added to define the systems in a hierarchical manner as given below:

System K: Propositional logic axioms + $L(\alpha \Rightarrow \beta) \Rightarrow (L\alpha \Rightarrow L\beta)$ (modal axiom K).

System D: System K + $(L\alpha \Rightarrow M\alpha)$ (modal axiom D) where $M\alpha = \neg L \neg \alpha$.

System T: System K + $(L\alpha \Rightarrow \alpha)$ (modal axiom T).

System S_4: System T + $(L\alpha \Rightarrow LL\alpha)$ (modal axiom S_4).

System B: System T + $(\alpha \Rightarrow LM\alpha)$ (modal axiom B).

System S_5: System T + $(ML\alpha \Rightarrow L\alpha)$ (modal axiom S_5).

There are two rules of inference:

$$MP \text{ (Modus Ponens): } \frac{\alpha,\ \alpha \Rightarrow \beta}{\beta}$$

and

$$N \text{ (Necessitation): } \frac{\alpha}{L\alpha}$$

for all wffs α, β.

The language or the set of all wffs shall be denoted by $\mathcal{L}_{ML}$. The usual semantics for modal systems is the Kripke semantics for which we refer to [18]. However, for our current purpose we present it with little modifications as below.

Kripke semantics: A Kripke frame is a pair $\mathcal{F}_{\mathcal{K}} = (U, \rho)$ where U is a non-empty set of worlds and ρ is a binary relation on U called the accessibility relation. A frame $\mathcal{F}_{\mathcal{K}} = (U, \rho)$ is said to be reflexive/symmetric/transitive if and only if ρ is so.

A Kripke model is a triple $\mathcal{M}_{\mathcal{K}} = (U, \rho, v)$ where (U, ρ) is a Kripke frame and $v : Prop \rightarrow 2^U$ is a valuation function from *Prop* (the set of all propositional variables) to the power set of U.

Given a Kripke frame $\mathcal{F}_{\mathcal{K}} = (U, \rho)$, the operation L_ρ: $2^U \rightarrow 2^U$ on the power set 2^U is defined by $L\rho(P) = \{u \in U : \rho(u) \subseteq P\}$ where $\rho(u) = \{w \in U : u\rho w\}$. The dual operation of L_ρ is defined by $M_\rho(P) = (L_\rho(P^c))^c = \{u \in U : \rho(u) \cap P \neq \emptyset\}$, where $(.)^c$ is the complement operation. $L_\rho(P)$ and $M_\rho(P)$ are respectively the lower and upper approximations of P as defined in Subsection 3.1 on page 257. We adopt the new notations to make the correspondence with the modal operators L and M transparent.

Definition 21. *The truth set of a modal formula* $\alpha \in \mathcal{L}_{ML}$ *in a Kripke model* $\mathcal{M}_{\mathcal{K}} = (U, \rho, v)$, *denoted by* $[\alpha]_{\mathcal{M}_{\mathcal{K}}}$, *is defined by:*

$[p]_{\mathcal{M}_{\mathcal{K}}} = v(p)$

$[\neg\alpha]_{\mathcal{M}_{\mathcal{K}}} = ([\alpha]_{\mathcal{M}_{\mathcal{K}}})^c$

$[\alpha \vee \beta]_{\mathcal{M}_{\mathcal{K}}} = [\alpha]_{\mathcal{M}_{\mathcal{K}}} \cup [\beta]_{\mathcal{M}_{\mathcal{K}}}$

$[L\alpha_{\mathcal{M}_{\mathcal{K}}} = L_{\rho}([\alpha]_{\mathcal{M}_{\mathcal{K}}})$.

A formula α is true (or satisfied) at u in a model $\mathcal{M}_{\mathcal{K}}$ *(notation:* $\mathcal{M}_{\mathcal{K}}, u \models_K \alpha$, *where the subscript K means 'Kripke') if* $u \in [\alpha]_{\mathcal{M}_{\mathcal{K}}}$. *A formula α is true in a model* $\mathcal{M}_{\mathcal{K}}$ *(notation:* $\mathcal{M}_{\mathcal{K}} \models_K \alpha$*) if* $[\alpha]_{\mathcal{M}_{\mathcal{K}}} = U$.

A formula α is valid at $u \in U$ *in a Kripke frame* $\mathcal{F}_{\mathcal{K}} = (U, \rho)$ *(notation:* $\mathcal{F}_{\mathcal{K}}, u \models_K \alpha$*) if α is true at u in every model* $\mathcal{M}_{\mathcal{K}} = (U, \rho, v)$.

A formula α is valid in a frame $\mathcal{F}_{\mathcal{K}}$ *(notation:* $\mathcal{F}_{\mathcal{K}} \models_K \alpha$*) if α is valid at each* $u \in U$ *in the frame* $\mathcal{F}_{\mathcal{K}}$.

For any modal system S, the set of all theorems in S is denoted by $Thm(S)$. A normal modal system S is characterized by a class of frames F if for any modal formula α, $\alpha \in Thm(S)$ if and only if α is valid in all frames in F. The following results are well known:

1. K is characterized by the class of all frames.
2. S_4 is characterized by the class of all S_4 frames.
3. B is characterized by the class of all reflexive and symmetric frames.

For details of proof, one can see [18] and [6].

We shall now present rough set semantics for the modal logic systems. In order to do that it is required to observe the rough set theoretic equivalents of modal properties expressed by modal formulas.

Rought set theoretic properties	*Corresponding modal properties*
$U = \underline{U}$	$\dfrac{\vdash \alpha}{\vdash L\alpha}$ (Rule N)
$\underline{P \cap Q} \subseteq \underline{P} \cap \underline{Q}$	$L(\alpha \wedge \beta) \Rightarrow (L\alpha \wedge L\beta)$
$\underline{P} \cap \underline{Q} \subseteq \underline{P \cap Q}$	$(L\alpha \wedge L\beta) \Rightarrow L(\alpha \wedge \beta)$
$P \subseteq Q$ implies $\underline{P} \subseteq \underline{Q}$	$\dfrac{\vdash \alpha \Rightarrow \beta}{\vdash L\alpha \Rightarrow L\beta}$
$\underline{P} \subseteq P$	$L\alpha \Rightarrow \alpha (T)$
$\underline{P} \subseteq \overline{P}$	$L\alpha \Rightarrow M\alpha (D)$
$P \subseteq \underline{\left(\overline{P}\right)}$	$\alpha \Rightarrow LM\alpha\ (B)$
$\underline{P} \subseteq \underline{\left(\underline{P}\right)}$	$L\alpha \Rightarrow LL\alpha\ (S_4)$
$\overline{\left(\underline{P}\right)} \subseteq \underline{P}$	$ML\alpha \Rightarrow L\alpha\ (S_5)$
$\underline{\left(P^c \cup Q\right)} \subseteq \left(\underline{P}\right)^c \cup \underline{Q}$	$L(\alpha \Rightarrow \beta) \Rightarrow (L\alpha \Rightarrow L\beta)\ (K)$

$\underline{P}$ and $\overline{P}$ are respectively the lower and upper approximations of the set P which are defined in the relational approach (Subsection 3.1) and covering based approach (Subsection 3.2 on page 259).

3.1 Relational approach

As mentioned in the introduction an approximation space in Pawlak's rough set theory is $\langle U, R\rangle$ where U is a non empty set and R is an equivalence relation on U. A pair of lower-upper approximations of any subset P of U is defined as

$$\underline{P}_R = \left\{u \in U : [u]_R \subseteq P\right\}$$

and

$$\overline{P}^R = \left\{u \in U : [u]_R \cap P \neq \emptyset\right\},$$

where $[u]_R = \{v \in U : uRv\}$. This notion has been generalized by taking an arbitrary relation ρ in lieu of the equivalence relation R and imposing conditions like reflexivity, symmetry and transitivity etc. gradually on ρ [52,48,41]. This has been done in two steps. In the first step, a granule ρ_u, for each $u \in U$, has been defined as $\rho_u = \{v \in U : u\rho v\}$. In the second step, for each subset P of U, lower approximation $\underline{P}_\rho$ and upper approximation $\overline{P}^\rho$ are defined by

$$\underline{P}_\rho = \{u \in U : \rho_u \subseteq P\} \ \left(= L_\rho(P)\right)$$

and

$$\overline{P}^\rho = \{u \in U : \rho_u \cap P \neq \emptyset\} \left(= M_\rho(P)\right).$$

With these definitions one can proceed towards their properties and depending on various properties (e.g. reflexivity, symmetry, transitivity, seriality and their various combinations) of the relation ρ various properties of the lower and upper approximations are obtained. The following Table 1 may be observed where the suffixes of ρ namely r, s and t or their combinations indicate that the relation is reflexive, symmetric and transitive respectively or their combinations. ρ_{ser} denotes a serial relation.

Table 1: Properties of relation based approximations

	ρ	ρ_r	ρ_s	ρ_t	ρ_{rs}	ρ_{rt}	ρ_{st}	ρ_{rst}	ρ_{ser}
Duality of $\underline{P}$, $\overline{P}$	Y	Y	Y	Y	Y	Y	Y	Y	Y
$\underline{\emptyset} = \emptyset$	N	N	N	N	Y	Y	N	Y	Y
$\emptyset = \overline{\emptyset}$	Y	Y	Y	Y	Y	Y	Y	Y	Y
$\underline{U} = U$	Y	Y	Y	Y	Y	Y	Y	Y	Y
$U = \overline{U}$	N	N	N	N	Y	Y	N	Y	Y
$\underline{P \cap Q} \subseteq \underline{P} \cap \underline{Q}$	Y	Y	Y	Y	Y	Y	Y	Y	Y
$\underline{P} \cap \underline{Q} \subseteq \underline{P \cap Q}$	Y	Y	Y	Y	Y	Y	Y	Y	Y
$\overline{P \cup Q} \subseteq \overline{P} \cup \overline{Q}$	Y	Y	Y	Y	Y	Y	Y	Y	Y
$\overline{P} \cup \overline{Q} \subseteq \overline{P \cup Q}$	Y	Y	Y	Y	Y	Y	Y	Y	Y
$P \subseteq Q$ implies $\underline{P} \subseteq \underline{Q}$	Y	Y	Y	Y	Y	Y	Y	Y	Y
$P \subseteq Q$ implies $\overline{P} \subseteq \overline{Q}$	Y	Y	Y	Y	Y	Y	Y	Y	Y
$\underline{P} \subseteq P$	N	Y	N	N	Y	Y	N	Y	N

$P \subseteq \overline{P}$	N	Y	N	N	Y	Y	N	Y	N
$\underline{P} \subseteq \overline{P}$	N	Y	N	N	Y	Y	N	Y	Y
$P \subseteq \underline{\left(\overline{P}\right)}$	N	N	Y	N	Y	N	Y	Y	N
$\overline{\left(\underline{P}\right)} \subseteq P$	N	N	Y	N	Y	N	Y	Y	N
$\underline{P} \subseteq \underline{\left(\underline{P}\right)}$	N	N	N	Y	N	Y	Y	Y	N
$\overline{\left(\overline{P}\right)} \subseteq \overline{P}$	N	N	N	Y	N	Y	Y	Y	N
$\overline{P} \subseteq \underline{\left(\overline{P}\right)}$	N	N	N	N	N	N	N	Y	N
$\overline{\left(\underline{P}\right)} \subseteq \underline{P}$	N	N	N	N	N	N	N	Y	N
$\underline{\left(P^c \cup Q\right)} \subseteq \left(\underline{P}\right)^c \cup \underline{Q}$	Y	Y	Y	Y	Y	Y	Y	Y	Y

The above table is nothing but the Kripke semantics after being translated in rough set semantics.

3.2 Covering based approach

For the definitions 22 to 28 given below in covering based approach we depend on [28, 32, 20, 14, 55, 53, 21, 39, 40, 41, 42, 10, 54, 33].

Definition 22. (*Covering of a set*): *Let U be a non empty set and* $C = \{U_i(\neq \emptyset) \subseteq U : i \in I\}$, *where I is an index set, is said to be a covering of U if*

$$\bigcup_{i \in I} U_i = U.$$

Definition 23. (*Covering approximation space*): *Let U be a non empty set and C be a covering of U. Then, the ordered pair* $\langle U, C \rangle$ *is called a covering approximation space.*

Definition 24. (*Friends of u*): *Let* $\langle U, C \rangle$ *be a covering approximation space. For each* $u \in U$, *Friends of u is defined by*

$$F^C(u) = \bigcup_{u \in U_i} U_i.$$

Definition 25. (*Neighborhood of u*): *Let $\langle U, C \rangle$ be a covering approximation space. For each $u \in U$, Neighborhood of u is defined by*

$$N^C(u) = \bigcap_{u \in U_i} U_i.$$

Definition 26. (*Friends' enemy of u*): *Let $\langle U, C \rangle$ be a covering approximation space. For each $u \in U$, Friends' enemy of u is defined by*

$$FE^C(u) = U - F^C(u)$$

Definition 27. (*Kernel of u*): *Let $\langle U, C \rangle$ be a covering approximation space. For each $u \in U$, Kernel of u is defined by*

$$K^C(u) = \{y \in U : \forall U_i (u \in U_i \Leftrightarrow y \in U_i)\}.$$

Let $\mathcal{P}^C = \{K^c(u) : u \in U\}$. Then, $\mathcal{P}^C$ is a partition of U and called partition generated by the covering C.

Definition 28. (*Minimal description and Maximal description of u*): *Let $\langle U, C \rangle$ be a covering approximation space. For each $u \in U$, Minimal description and Maximal description of u are defined respectively as*

$$md^{\mathcal{C}}(u) = \{U_i \in \mathcal{C} : u \in U_i \textit{ and } \forall U_j \in \mathcal{C}(u \in U_j \subseteq U_i \textit{ implies } U_j = U_i)\}.$$

and

$$Md^{\mathcal{C}}(u) = \{U_i \in \mathcal{C} : u \in U_i \text{ and } \forall\, U_j \in \mathcal{C}(U_j \supseteq U_i \textit{ implies } U_j = U_i)\}.$$

It is to be noted that both $md^C(u)$ and $Md^C(u)$ are subsets of 2^U, power set of U, while others are subsets of U.

Various Types of Lower and Upper Approximations

There are many lower-upper approximations in different literature based on covering cases. Some of them are dual approximations with respect to the set theoretic complementation while others are not so, called non-dual approximations. For our purposes, some of them (both dual and non-dual) available in [32, 39, 51, 20, 47, 55, 9, 14, 28, 33, 49, 53, 55, 41,21,10,42,19] are presented below.

$$\underline{P_1}(P) = \{u : F^C(u) \subseteq P\}.$$
$$\overline{P_1}(P) = \cup\{U_i : U_i \cap P \neq \emptyset\}.$$

$$\underline{P_2}(P) = \cup\{F^C(u) : F^C(u) \subseteq P\}.$$
$$\overline{P_2}(P) = \{z : \forall y(z \in F^C(y) \Rightarrow F^C(y) \cap P \neq \emptyset)\}.$$

$$\underline{P_3}(P) = \cup\{U_i : U_i \subseteq P\}.$$
$$\overline{P_3}(P) = \{z : \forall U_i(z \in U_i \Rightarrow U_i \cap P \neq \emptyset)\}.$$

$$\underline{P_4}(P) = \cup\{K^C(u) : K^C(u) \subseteq P\}.$$
$$\overline{P_4}(P) = \cup\{K^C(u) : K^C(u) \cap P \neq \emptyset\}.$$

$$\underline{C_1}(P) = \cup\{U_i : U_i \subseteq P\}.$$
$$\overline{C_1}(P) = \left(\underline{(A^c)}\right)^c = \cap\{U_i^c : U_i \cap P = \emptyset\}.$$

$$\underline{C_2}(P) = \{u \in U : N^C(u) \subseteq P\}.$$
$$\overline{C_2}(P) = \{u \in U : N^C(u) \cap P \neq \emptyset\}.$$

$$\underline{C_3}(P) = \{u \in U : \exists x(x \in N^C(u) \text{ and } \mathrm{N}^C(x) \subseteq P)\}.$$
$$\overline{C_3}(P) = \{u \in U : \forall x(x \in N^C(u) \Rightarrow N^C(x) \cap P \neq \emptyset)\}.$$

$$\underline{C_4}(P) = \{u \in U : \forall x(u \in N^C(x) \Rightarrow \mathrm{N}^C(x) \subseteq P)\}.$$
$$\overline{C_4}(P) = \cup\{N^C(u) : N^C(u) \cap P \neq \emptyset\}.$$

$$\underline{C_5}(P) = \{u \in U : \forall x(u \in N^C(x) \Rightarrow x \in P)\}.$$
$$\overline{C_5}(P) = \cup\{N^C(u) : u \in P\}.$$

With the same lower approximation there are a few different upper approximations.

$$\underline{C_*}(P)=\underline{C_-}(P)=\underline{C_\#}(P)=\underline{C_@}(P)=\underline{C_+}(P)=\underline{C_\%}(P)=\underline{P_3}(P)=\cup\{U_i\in C:U_i\subseteq P\}.$$

But the corresponding upper approximations are as follows.

$$\overline{C_*}(P)=\underline{C_*}(P)\cup\left\{T:T\in md^C(x)\text{ and }x\in P-\underline{C_*}(P)\right\}.$$

$$\overline{C_-}(P)=\cup\{U_i:U_i\cap P\neq\emptyset\}.$$

$$\overline{C_\#}(P)=\cup\left\{T:T\in md^C(x)\text{ and }x\in P\right\}.$$

$$\overline{C_@}(P)=\underline{C_@}(P)\cup\left\{U_i:U_i\cap\left(P-\underline{C_@}(P)\right)\neq\emptyset\right\}.$$

$$\overline{C_+}(P)=\underline{C_+}(P)\cup\left\{N^C(u):u\in\left(P-\underline{C_+}(P)\right)\neq\emptyset\right\}.$$

$$\overline{C_\%}(P)=\underline{C_\%}(P)\cup\left(\cup\left\{F^C(u):u\in FE^C(x),\ x\in\left(P-\underline{C_\%}(P)\right)\right\}\right)^c.$$

Two other types of lower and upper approximations are defined with the help of covering.

(1) Let, $Gr_*(P)=\cup\{U_i : U_i\subseteq P\}\equiv\underline{P}_3(P)$.

This is taken as lower approximation of P and is denoted by $\underline{C_{Gr}}(P)$.

Let, $Gr^*(P)=\cup\{U_i:U_i\cap P\neq\emptyset\}\equiv\overline{P_1}(P)$.

The upper approximation is defined by $\overline{C^{Gr}}(P) = Gr^*(P) — NEG_{Gr}(P)$, where $NEG_{Gr}(P)=\underline{C_{Gr}}(P^c)$.

(2) A set D is said to be definable if and only if there exists a set A ($\subseteq U$) such that $D = U_{x\in A}N^C(x)$. Let $\mathcal{D} = \{D \subseteq U : D \text{ is definable}\}$. $\underline{C_t}, \overline{C_t} : 2^U \to 2^U$ are such that $\underline{C_t}(A) = \cup\{D \in \mathcal{D} : D \subseteq A\}$ and $\overline{C_t}(A) = \cup\{D \in \mathcal{D} : A \subseteq D\}$

It may be observed that $\cup\{D \in \mathcal{D} : D \subseteq A\} = \cup\{N^C(x) : N^C(x) \subseteq A\} = \{x \in U : N^C(x) \subseteq A\} = \underline{C_t}(A)$ and $\cup\{D \in \mathcal{D} : A \subseteq D\} = \cup\{N^C(x) : x \in A\} = \overline{C_t}(A)$.

The properties of various lower and upper approximations have been summarized in Table 2 [42].

Table 2: Properties of covering based approximations

	P_1	P_2	P_3	P_4	C_1	C_2	C_3	C_4	C_5	C_{Gr}	C_*	C_-	$C_\#$	$C_@$	C_+	$C_\%$	C_t
Duality of $\underline{P}$, $\overline{P}$	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	N	N	N	N	N	N	N
$\underline{\emptyset} = \emptyset = \overline{\emptyset}$	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
$\underline{U} = U = \overline{U}$	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
$\underline{P \cap Q} \subseteq \underline{P} \cap \underline{Q}$	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
$\underline{P} \cap \underline{Q} \subseteq \underline{P \cap Q}$	Y	N	N	Y	N	Y	N	Y	Y	N	N	N	N	N	N	N	Y
$\overline{P \cup Q} \subseteq \overline{P} \cup \overline{Q}$	Y	N	N	Y	N	Y	N	Y	Y	N	N	Y	N	Y	Y	N	Y
$\overline{P} \cup \overline{Q} \subseteq \overline{P \cup Q}$	Y	Y	Y	Y	Y	Y	N	Y	Y	Y	Y	Y	Y	Y	Y	N	Y
$P \subseteq Q$ implies $\underline{P} \subseteq \underline{Q}$	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
$P \subseteq Q$ implies $\overline{P} \subseteq \overline{Q}$	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	N	Y	Y	N	Y	Y	Y
$\underline{P} \subseteq P$	Y	Y	Y	Y	Y	Y	N	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
$P \subseteq \overline{P}$	Y	Y	Y	Y	Y	Y	N	Y	Y	Y	Y	Y	Y	Y	Y	N	Y
$\underline{P} \subseteq \overline{P}$	Y	Y	Y	Y	Y	Y	N	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
$P \subseteq \underline{\left(\overline{P}\right)}$	Y	N	N	Y	N	N	N	Y	N	N	Y	Y	Y	Y	N	N	Y
$\overline{\left(\underline{P}\right)} \subseteq P$	Y	N	N	Y	N	N	N	Y	N	N	N	N	Y	Y	Y	N	Y
$\underline{P} \subseteq \underline{\left(\underline{P}\right)}$	N	Y	Y	Y	Y	Y	N	N	Y	Y	Y	Y	Y	Y	Y	Y	Y
$\overline{\left(\overline{P}\right)} \subseteq \overline{P}$	N	Y	Y	Y	Y	Y	N	N	Y	Y	Y	N	N	Y	Y	Y	Y

$\overline{P} \subseteq \left(\overline{P}\right)$	N	N	N	Y	N	N	N	N	N	N	Y	Y	Y	Y	N	N	Y
$\overline{\left(\underline{P}\right)} \subseteq \underline{P}$	N	N	N	Y	N	N	N	N	N	N	N	N	Y	Y	Y	N	Y
$\underline{\left(P^c \cup Q\right)} \subseteq \left(\underline{P}\right)^c \cup \underline{Q}$	Y	N	N	Y	N	Y	N	Y	Y	N	N	N	N	N	N	N	Y

A few points should be noted here. First that the properties taken are not independent. Second, which is not properly noticed always, is that some of the approximations, though defined differently, are in fact the same. Such a case is the following.

$$\overline{P_3}(A) = \overline{C_1}(A) = \overline{C_{Gr}}(A), \text{ for all } A \subseteq U.$$

Third, covering systems P_1 to C_{Gr} are dual with respect to the lower and upper approximation operators while C_* to C_t are non-dual systems.

Observation 1. In Table 2, two pairs $\underline{P_1}$, $\underline{C_4}$ and $\underline{C_2}$, $\underline{C_5}$ have identical columns. Still they are different operators as shown by examples given below.

Example 3. [42] Let $U = \{1, 2, 3, 4, 5, 6\}$ and $C = \{U_1, U_2, U_3, U_4\}$ where $U_1 = \{1, 2\}$, $U_2 = \{2, 3, 4\}$, $U_3 = \{4, 5\}$ and $U_4 = \{6\}$. Now, $F^C(1) = \cup_{1 \in U_i} U_i = \{1, 2\}$, $F^C(2) = \{1, 2, 3, 4\}$, $F^C(3) = \{2, 3, 4\}$, $F^C(4) = \{2, 3, 4, 5\}$, $F^C(5) = \{4, 5\}$, $F^C(6) = \{6\}$ and $N^C(1) = \{1, 2\}$, $N^C(2) = \{2\}$, $N^C(3) = \{2,3,4\}$, $N^c(4) = \{4\}$, $N^C(5) = \{4, 5\}$, $N^C(6) = \{6\}$. Let $Q = \{4, 5\}$. Then, $\underline{P_1}(Q) = \{5\}$ and $\underline{C_4}(Q) = \emptyset$.

Thus, $\underline{P_1}$ and $\underline{C_4}$ are different.

Example 4. [42] Let $U = \{1, 2, 3\}$ and $C = \{U_1, U_2\}$ where $U_1 = \{1, 2\}$, $U_2 = \{2, 3\}$. Now, $N^C(1) = \{1, 2\}$, $N^C(2) = \{2\}$, $N^C(3) = \{2, 3\}$. Let $Q = \{1, 2\}$. Then, $\underline{C_2}(Q) = \{1, 2\}$ and $\underline{C_5}(Q) = \{1\}$. Thus, $\underline{C_2}$ and $\underline{C_5}$ are different.

We now present a table 3 [42] of dual systems, covering as well as relation based. In this table only the standard modal axioms are considered.

Table 3: Table of dual systems.

	P_1	P_2	P_3	P_4	C_1	C_2	C_3	C_4	C_5	C_{Gr}	R	R_r	R_s	R_t	R_{rs}	R_{rt}	R_{st}	R_{rst}
K	Y	N	N	Y	N	Y	N	Y	Y	N	Y	Y	Y	Y	Y	Y	Y	Y
D	Y	Y	Y	Y	Y	Y	N	Y	Y	Y	N	Y	N	N	Y	Y	N	Y
T	Y	Y	Y	Y	Y	Y	N	Y	Y	Y	N	Y	N	N	Y	Y	N	Y
B	Y	N	N	Y	N	N	N	Y	N	N	N	N	Y	N	Y	N	Y	Y
S_4	N	Y	Y	Y	Y	Y	N	N	Y	Y	N	N	N	Y	N	Y	Y	Y
S_5	N	N	N	Y	N	N	N	N	N	N	N	N	N	N	N	N	N	Y

Let us now focus only on the systems which possess K. Then depending on the identity of the columns below, the systems are clustered in the following groups: $\{P_1, C_4, R_{rs}\}$, $\{P_4, R_{rst}\}$ and $\{C_2, C_5, R_{rt}\}$. Identity of P_4 and R_{rst} was evident right from the beginning of Rough Set theory. From the other two groups we can say that P_1 and C_4 are at least modal system B (because of the presence of R_{rs} in the group) and not S_5. Similarly, systems C_2 and C_5 are at least S_4 and not S_5. It has been proved in [22] that P_1 and C_4 are exactly system B and C_2 and C5 are exactly system S_4.

Covering semantics:

Definition 29. *A covering frame is a pair $\mathcal{F}_{\mathcal{C}} = (U, \mathcal{C})$ where U is a non-empty set and $\mathcal{C}$ is a covering of U.*

A covering model is a triple $\mathcal{M}_{\mathcal{C}} = (U, \mathcal{C}, v)$ where $(U, \mathcal{C})$ is a covering frame and $v : Prop \to 2^U$ is a valuation.

Covering semantics for modal logic differs from the Kripke semantics only in the interpretation of modalities *L* and *M*.

Definition 30. *The truth set of a modal formula $\alpha \in \mathcal{L}_{ML}$ in* a *covering model $\mathcal{M}_{\mathcal{C}} = (U, \mathcal{C}, v)$ under λ semantics where $\lambda \in \{P_1, C_4, C_2, C_5\}$, denoted by $[\alpha]^{\lambda}_{\mathcal{M}_{\mathcal{C}}}$, is defined by:*

$$[p]^{\lambda}_{\mathcal{M}_{\mathcal{C}}} = v(p)$$

$$[\neg\alpha]^{\lambda}_{\mathcal{M}_{\mathcal{C}}} = ([\alpha]^{\lambda}_{\mathcal{M}_{\mathcal{C}}})^c$$

$$[\alpha \vee \beta]^{\lambda}_{\mathcal{M}_{\mathcal{C}}} = [\alpha]^{\lambda}_{\mathcal{M}_{\mathcal{C}}} \cup [\beta]^{\lambda}_{\mathcal{M}_{\mathcal{C}}}$$

$$[L\alpha]^{\lambda}_{\mathcal{M}_{\mathcal{C}}} = \underline{\lambda}([\alpha]^{\lambda}_{\mathcal{M}_{\mathcal{C}}}).$$

A formula α is true (or satisfied) at u in a covering model $\mathcal{M}_{\mathcal{C}}$ (notation: $\mathcal{M}_{\mathcal{C}}, u \models_{\lambda} \alpha$) if $u \in [\alpha]^{\lambda}_{\mathcal{M}\mathcal{C}}$.

A formula α is true in a model $\mathcal{M}_{\mathcal{C}}$ (notation: $\mathcal{M}_{\mathcal{C}} \models_{\lambda} \alpha$) if $[\alpha]^{\lambda}_{\mathcal{M}_{\mathcal{C}}} = U$.

A formula α is valid at $u \in U$ in a covering frame $\mathcal{F}_{\mathcal{C}} = (U, \mathcal{C})$ (notation: $\mathcal{F}_{\mathcal{C}}, u \models_{\lambda} \alpha$) if α is true at u in every model $\mathcal{M}_{\mathcal{C}} = (U, \mathcal{C}, v)$. A formula α is valid in a frame $\mathcal{F}_{\mathcal{C}}$ (notation: $\mathcal{F}_{\mathcal{C}} \models_{\lambda} \alpha$) if α is valid at each $u \in U$ in the frame $\mathcal{F}_{\mathcal{C}}$.

Let τ(λ) be the set of all tautologies in λ semantics, that is, $\tau(\lambda) = \{\alpha \in \mathcal{L}_{ML} : \mathcal{F}_{\mathcal{C}} \models_{\lambda} \alpha$ for any covering frame $\mathcal{F}_{\mathcal{C}}\}$.

P_1 and C_4 semantics:

With the above general definition of covering semantics we proceed for these particular covering systems. Thus for modal operators, P semantic clauses in a covering model $\mathcal{M}_{\mathcal{C}} = (U, \mathcal{C}, v)$ are the following:

$\mathcal{M}_{\mathcal{C}}, u \models_{P_1} L\alpha$ if and only if for all $U_i \in \mathcal{C}$ and for all $w \in U(u, w \in U_i$ implies $\mathcal{M}_{\mathcal{C}}, w \models_{P_1} \alpha)$.

$\mathcal{M}_{\mathcal{C}}, u \models_{P_1} M\alpha$ if and only if there exists $U_i \in \mathcal{C}$ such that for some $w \in U(u, w \in U_i$ and $\mathcal{M}_{\mathcal{C}}, w \models_{P_1} \alpha)$.

The strategy that has been used [22] to prove $\tau(P_1) = Thm(B)$ is by reducing the P_1 semantics to the Kripke semantics for the modal logic B.

Definition 31. *For a reflexive and symmetric Kripke frame $\mathcal{F}_{\mathcal{K}} = (U, \rho)$, the P_1 lifting of $\mathcal{F}_{\mathcal{K}}$ is defined by the structure $\mathcal{F}_{\mathcal{C}}{}^{P_1} = (U, \mathcal{C}_{\rho}{}^{P_1})$ where $\mathcal{C}_{\rho}{}^{P_1} = \{\{u, w\} \subseteq U : u\rho w\}$*

The function $N_{\rho}{}^{P_1} : U \rightarrow 2^U$ is defined by $N_{\rho}{}^{P_1}(u) = \cup\{\{u, w\} \in \mathcal{C}_{\rho}{}^{P_1} : w \in U\}$.

Proposition 2. *Let $\mathcal{F}_{\mathcal{C}}{}^{P_1} = (U, \mathcal{C}_{\rho}{}^{P_1})$ be the P_1 lifting of a reflexive and symmetric Kripke frame $\mathcal{F}_{\mathcal{K}} = (U, \rho)$. Then the following hold:*

1. *$\mathcal{C}_{\rho}{}^{P_1}$ is a covering of U.*
2. *For all $u \in U$, $N_{\rho}{}^{P_1}(u) = \rho(u)$.*

Thus, for any reflexive and symmetric Kripke model $\mathcal{M}_{\mathcal{K}} = (U, \rho, v)$, we have the covering model (P_1 lifting of $\mathcal{M}_{\mathcal{K}}$) $\mathcal{M}_{\mathcal{C}}{}^{P_1} = (U, \mathcal{C}_{\rho}{}^{P_1}, v)$.

Lemma 1. *Let $\mathcal{F}_{\mathcal{K}} = (U, \rho)$ and $\mathcal{M}_{\mathcal{K}} = (U, \rho, v)$ be any reflexive and symmetric Kripke frame and model. Then for any formula α and for any $u \in U$, the following hold:*

1. $\mathcal{M}_{\mathcal{K}}, u \models_K \alpha$ *if and only if* $\mathcal{M}_{\mathcal{C}}{}^{P_1}, \models_{P_1} \alpha$.
2. $\mathcal{M}_{\mathcal{K}} \models_K \alpha$ *if and only if* $\mathcal{M}_{\mathcal{C}}{}^{P_1} \models_{P_1} \alpha$.
3. *$\mathcal{F}_{\mathcal{K}}, u \models_K \alpha$ if and only if $\mathcal{F}_{\mathcal{C}}{}^{P_1}, u \models_{P_1} \alpha$*
4. *$\mathcal{F}_{\mathcal{K}} \models_K \alpha$ if and only if $\mathcal{F}_{\mathcal{C}}{}^{P_1} \models_{P_1} \alpha$*

Theorem 6. $\tau(P_1) = Thm(B)$.

We now discuss about the logic of C_4 semantics. It is also exactly the modal logic B. For modal operators, C_4 semantic clauses in a covering model $\mathcal{M}_{\mathcal{C}} = (U, \mathcal{C}, v)$ are the following:

$\mathcal{M}_{\mathcal{C}}, u \models_{C_4} L\alpha$ if and only if for all $x, w \in U(u, w \in N^{C}(x)$ implies $\mathcal{M}_{\mathcal{C}}, w \models_{C_4} \alpha)$.

$\mathcal{M}_{\mathcal{C}}, u \models_{C_4} M\alpha$ if and only if there exist $x, w \in U(u, w \in N^{C}(x)$ and $\mathcal{M}_{\mathcal{C}}, w \models_{C_4} \alpha)$.

Definition 32. *For a reflexive and symmetric Kripke frame $\mathcal{F}_{\mathcal{K}} = (U, \rho)$, the C_4 lifting of $\mathcal{F}_{\mathcal{K}}$ is defined by the structure $\mathcal{F}_{\mathcal{C}}^{C_4} = (U, \mathcal{C}_{\rho}^{C_4})$ where $\mathcal{C}_{\rho}^{C_4} = \{\rho(u) : u \in U\}$*

The function $N_{\rho}^{C_4} : U \to 2^U$ is defined by $N_{\rho}^{C_4}(u) = \cup\{\rho(w) \in \mathcal{C}_{\rho}^{C_4} : u \in \rho(w)\}$.

As ρ is reflexive, $\mathcal{C}_{\rho}^{C_4}$ is a covering of U.

Thus, for any reflexive and symmetric Kripke model $\mathcal{M}_{\mathcal{K}} = (U, \rho, v)$, we have the covering model (C_4 lifting of $\mathcal{M}_{\mathcal{K}}$) $\mathcal{M}_{\mathcal{C}}^{C_4} = (U, \mathcal{C}_{\rho}^{C_4}, v)$.

Lemma 2. *Let $\mathcal{F}_{\mathcal{K}} = (U, \rho)$ and $\mathcal{M}_{\mathcal{K}} = (U, \rho, v)$ be any reflexive and symmetric Kripke frame and model. Then for any formula α and for any $u \in U$, the following hold:*

1. $\mathcal{M}_{\mathcal{K}}, u \models_K \alpha$ *if and only if* $\mathcal{M}_{\mathcal{C}}^{C_4}, u \models_{C_4} \alpha$.
2. $\mathcal{M}_{\mathcal{K}} \models_K \alpha$ *if and only if* $\mathcal{M}_{\mathcal{C}}^{C_4} \models_{C_4} \alpha$.
3. $\mathcal{F}_{\mathcal{K}}$, u $\models_K \alpha$ *if and only if* $\mathcal{F}_{\mathcal{C}}^{C_4}, u \models_{C_4} \alpha$
4. $\mathcal{F}_{\mathcal{K}} \models_K \alpha$ *if and only if* $\mathcal{F}_{\mathcal{C}}^{C_4} \models_{C_4} \alpha$

Theorem 7. $\tau(C4) = Thm(B)$.

From Theorem 6 and Theorem 7, it follows that $\tau(P_1) = Thm(B) = \tau(C_4)$.

C_2 and C_5 semantics:

The C_2-semantic clauses for modal operators in a covering model $\mathcal{M}_{\mathcal{C}} = (U, \mathcal{C}, v)$ are as follows:

$\mathcal{M}_{\mathcal{C}}, u \models_{C_2} L\alpha$ if and only if for all $w \in U(w \in N^{C}(u)$ implies $\mathcal{M}_{\mathcal{C}}, w \models_{C_2} \alpha)$

$\mathcal{M}_{\mathcal{C}}, u \models_{C_2} M\alpha$ if and only if there exists $w \in U(w \in N^{C}(u)$ and $\mathcal{M}_{\mathcal{C}}, w \models_{C_2} \alpha)$

Definition 33. *For an S_4 frame $\mathcal{F}_{\mathcal{K}} = (U, \rho)$, the C_2 lifting of $\mathcal{F}_{\mathcal{K}}$ is defined by the structure $\mathcal{F}_{\mathcal{C}}^{C_2} = (U, \mathcal{C}_{\rho}^{C_2})$ where $\mathcal{C}_{\rho}^{C_2} = \{\rho(u) : u \in U\}$. The function $N_{\rho}^{C_2} : U \to 2^U$ is defined by $N_{\rho}^{C_2}(u) = \cap\{\rho(w) \in \mathcal{C}_{\rho}^{C_2} : u \in \rho(w)\}$.*

Proposition 3. *Let $\mathcal{F}_{\mathcal{C}}^{C_2} = (U, \mathcal{C}_{\rho}^{C_2})$ be the C_2 lifting of an S_4 frame $\mathcal{F}_{\mathcal{K}} = (U, \rho)$. Then the following hold:*

1. $\mathcal{C}_\rho^{C2}$ *is a covering of U.*
2. *For all* $u \in U$, $N_\rho^{C2}(u) = \rho(u)$.

Thus, for any S_4 model $\mathcal{M}_\mathcal{K} = (U, \rho, v)$, we have the covering model $\mathcal{M}_\mathcal{C}^{C2} = (U, \mathcal{C}_\rho^{C2}, v)$.

Lemma 3. *Let* $\mathcal{F}_\mathcal{K} = (U, \rho)$ *and* $\mathcal{M}_\mathcal{K} = (U, \rho, v)$ *be any* S_4 *frame and model. Then for any formula* α *and for any* $u \in U$, *the following hold:*

1. $\mathcal{M}_\mathcal{K}, u \models_K \alpha$ *if and only if* $\mathcal{M}_\mathcal{C}^{C2}, u \models_{C_2} \alpha$.
2. $\mathcal{M}_\mathcal{K} \models_K \alpha$ *if and only if* $\mathcal{M}_\mathcal{C}^{C2} \models_{C_2} \alpha$.
3. $\mathcal{F}_\mathcal{K}$, u $\models_K \alpha$ *if and only if* $\mathcal{F}_\mathcal{C}^{C2}, u \models_{C_2} \alpha$
4. $\mathcal{F}_\mathcal{K} \models_K \alpha$ *if and only if* $\mathcal{F}_\mathcal{C}^{C2} \models_{C_2} \alpha$

Theorem 8. $\tau(C_2) = Thm(S_4)$.

We now present the C_5 semantics. For modal operators, semantic clauses are the following:

$\mathcal{M}_\mathcal{C}, u \models_{C_5} L\alpha$ if and only if for all $w \in U(u \in N^C(w)$ implies $\mathcal{M}_\mathcal{C}, w \models_{C_5} \alpha)$.

$\mathcal{M}_\mathcal{C}, u \models_{C_5} M\alpha$ if and only if there exists $w \in U(u \in N^C(w)$ and $\mathcal{M}_\mathcal{C}, w \models_{C_5} \alpha)$

It has been shown that the logic of C_5 semantics is also the modal logic S_4.

Definition 34. *For an* S_4 *frame* $\mathcal{F}_\mathcal{K} = (U, \rho)$, *the* C_5 *lifting of* $\mathcal{F}_\mathcal{K}$ *is defined by the structure* $\mathcal{F}_\mathcal{C}^{C5} = (U, \mathcal{C}_\rho^{C5})$ *where* $\mathcal{C}_\rho^{C5} = \{\rho^{-1}(u) : u \in U\}$ *and* $\rho^{-1}(u) = \{w \in U : w\rho u\}$.

Moreover, the function $N_\rho^{C5} : U \to 2^U$ *is defined by* $N_\rho^{C5}(u) = \bigcap\{\rho^{-1}(w) \in \mathcal{C}_\rho^{C5} : u \in \rho^{-1}(w)\}$.

Proposition 4. *Let* $\mathcal{F}_\mathcal{C}^{C5} = (U, \mathcal{C}_\rho^{C5})$ *be the* C_5 *lifting of an* S_4 *frame* $\mathcal{F}_\mathcal{K} = (U, \rho)$. *Then the following hold:*

1. $\mathcal{C}_\rho^{C5}$ *is a covering of U.*
2. *For all* $u \in U$, $N_\rho^{C5}(u) = \rho^{-1}(u)$.

Thus, for any S_4 model $\mathcal{M}_\mathcal{K} = (U, \rho, v)$, we have the covering model (C_5 lifting of $\mathcal{M}_\mathcal{K}$) $\mathcal{M}_C^{C5} = (U, \mathcal{C}_\rho^{C5}, v)$.

Lemma 4. *Let* $\mathcal{F}_\mathcal{K} = (U, \rho)$ *and* $\mathcal{M}_\mathcal{K} = (U, \rho, v)$ *be any* S_4 *frame and model. Then for any modal formula* α *and for any* $u \in U$, *the following hold:*

1. $\mathcal{M}_{\mathcal{K}}, u \models_K \alpha$ *if and only if* $\mathcal{M}_{\mathcal{C}}^{C5}, u \models_{C_5} \alpha$.
2. $\mathcal{M}_{\mathcal{K}} \models_K \alpha$ *if and only if* $\mathcal{M}_{\mathcal{C}}^{C5} \models_{C_5} \alpha$.
3. $\mathcal{F}_{\mathcal{K}}$, u $\models_K \alpha$ *if and only if* $\mathcal{F}_{\mathcal{C}}^{C5}, u \models_{C_5} \alpha$
4. $\mathcal{F}_{\mathcal{K}} \models_K \alpha$ *if and only if* $\mathcal{F}_{\mathcal{C}}^{C5} \models_{C_5} \alpha$

Theorem 9. $\tau(C_5) = Thm(S_4)$.

From Theorem 8 and Theorem 9, it follows that $\tau(C_2) = Thm(S_4) = \tau(C_5)$.

Thus, the modal system B captures P_1 and C_4 semantics and for C_2 and C_5 semantics, S_4 serves the purpose.

In [25] a modal system for C_1 semantics is obtained. Moreover, the technique used in [25] to develop the modal system for C_1 semantics is different from the method adopted in [22]. Completeness theorem for C_1 semantics has been proved by constructing a canonical covering model.

We now present the C_1 semantics. For modal operators, semantics clauses are the following:

$\mathcal{M}_{\mathcal{C}}, u \models_{C_1} L\alpha$ if and only if there exists $U_i \in C$ such that $u \in U_i$ and $\mathcal{M}_{\mathcal{C}}, w \models_{C_1} \alpha$, for all $w \in U_i$.

$\mathcal{M}_{\mathcal{C}}, u \models_{C_1} M\alpha$ if and only if for all $U_i \in C$, either $u \in U_i$ or there exists $w \in U_i$ such that $\mathcal{M}_{\mathcal{C}}, w \models_{C_1} \alpha$.

A new modal system ML_{C_1} is defined which consists of the following axioms and rules of inference.

Axioms:

PC_A: All axioms of classical propositional logic,

M: $L(\alpha \wedge \beta) \Rightarrow (L\alpha \wedge L\beta)$,

Top: $L\top$, where $\top$ is a propositional constant in the alphabet of the language of ML_{C_1} ,

T: $L\alpha \Rightarrow \alpha$,

S_4: $L\alpha \Rightarrow LL\alpha$,

Rules:

$$\text{MP}: \frac{\alpha,\ \alpha \Rightarrow \beta}{\beta},$$

$$\text{RE}: \frac{\alpha \Leftrightarrow \beta}{L\alpha \Leftrightarrow L\beta}.$$

It is to be noted that the inference rules N and RM are derivable in ML_{C_1}, where N, RM are

$$\text{N}: \frac{\alpha}{L\alpha},$$

$$\text{RM}: \frac{\alpha \Rightarrow \beta}{L\alpha \Rightarrow L\beta}.$$

One can easily verify the following soundness theorem by proving the validity of the axioms and the inference rules. $\vdash_{ML_{C_1}} \alpha$ means α is a theorem of ML_{C_1}

Theorem 10. (*Soundness*)*: For each wff* α, *if* $\vdash_{ML_{C_1}} \alpha$ *then* $\models_{C_1} \alpha$.

We now present a sketchy proof of the corresponding completeness theorem available in [25]. Only the modal points are presented here. Recall the notion of maximal consistent set [6]. The following notion of canonical covering based on maximal consistent sets plays a key role in the proof. The notation M_{cs} is used to denote the set of all ML_{C_1}-maximal consistent sets. Further, let $\alpha_1, \alpha_2, \ldots$ be an enumeration of all the wffs of the language of ML_{C_1}.

Definition 35. (*Canonical Covering Model*)*: The canonical covering model is defined as the tuple* $\mathcal{M}_{CC} = (M_{cs}, \mathcal{C}', v')$, *where*

— $\mathcal{C}'_i = \{\Delta \in M_{cs} : \alpha_i \wedge L\alpha_i \in \Delta\}, \forall_i \in \mathbb{N}$;

— $\mathcal{C}' = \{\mathcal{C}'_i : i \in \mathbb{N}\}$;

— $v'(p) = \{\Delta \in M_{cs} : p \in \Delta\}$.

It is to be noted that $\mathcal{C}'_i$ may be an empty set for some $i \in \mathbb{N}$.

Lemma 5. $(M_{cs}, \mathcal{C})$ *is a covering space.*

Lemma 6. (*Lindenbaum's Lemma*): *Let Δ be an ML_{C_1}-consistent set of wffs. Then there exists an ML_{C_1}-maximal consistent set Δ^+ containing Δ.*

Lemma 7. (*Existence Lemma*): *Let $\Delta \in ML_{C_1}$. The following hold.*

1. *If $L\alpha \in \Delta$ then there exists a $\mathcal{C}'_i \in \mathcal{C}'$ such that $\Delta \in \mathcal{C}'_i$ and $\alpha \in \Delta'$ for all $\Delta' \in \mathcal{C}'_i$.*
2. *If $M\alpha \in \Delta$ then for all $\mathcal{C}'_i \in \mathcal{C}'$, either $\Delta \notin \mathcal{C}'_i$ or there exists a $\Delta' \in \mathcal{C}'_i$ such that $\alpha \in \Delta'$.*

Lemma 8. (*Truth Lemma*): *For any wff α and $\Delta \in M_{cs}$, $\alpha \in \Delta$ if and only if $\mathcal{M}_{CC}, \Delta \models_{C_1} \alpha$.*

Theorem 11. (*Completeness Theorem*): *For any wff α, if $\models_{C_1} \alpha$ then $\vdash_{ML_{C_1}} \alpha$.*

Similarly it can be shown that for $\sigma \in \{P_3, C_{Gr}\}$, $\models_\sigma \alpha$ if and only $\vdash_{ML_{C_1}} \alpha$.

Thus modal systems corresponding to groups of covering systems $\{P_1, C_4\}$, $\{P_4\}$, $\{C_2, C_5\}$, $\{C_1, P_3, C_{Gr}\}$ are obtained. Besides, a modal system (which is a bi-modal one) corresponding to the non-dual covering system *Ct* was proposed in [17] and reported in [25]. For the other covering systems the problem is open.

3.3 Rough Consequence Logics and Approximate Reasoning

Another direction of research in logic arising from rough set studies is what is known as rough consequence logics. This constitutes a cluster of logics which are generalization of modal logics and are based upon rough modus ponens (**RMP**) rules. The first paper in this direction is [13]. Afterwards Martin Bunder published a paper [11] which generated momentum and the main work on this topic is a joint paper [12] published in 2008. Samanta [38] further contributed in this area. The idea in its most generality is to graft a logic on top of a modal system *S* with the help of the new rules of inference RMP.

Let *S* be a modal system with $\vdash_S$ as its consequence relation. New logic systems S_R are then defined using rough consequence relation $|\!\!\vdash_R$ by the following axioms. For all sets Γ of wffs and a wff α,

1. If $\vdash_S \alpha$ then $\Gamma \,|\!\!\vdash_R \alpha$.
2. $\{\alpha\} \,|\!\!\vdash_R \alpha$.

3. If $\Gamma \Vdash_R \alpha$ then $\Gamma \cup \Delta \Vdash_R \alpha$.
4. RMP may be applied to obtain a step in the derivation.

A group of rules fall under the category RMP viz.

$$\frac{\Gamma \Vdash_R \alpha,\ \Gamma \Vdash_R \beta \Rightarrow \delta,\ \vdash_S F(\alpha, \beta)}{\Gamma \Vdash_R \delta}$$

where $F(\alpha, \beta)$ is one of the following list of well formed formulas.

(*a*) $L\alpha \Rightarrow L\beta$ (*b*) $L\alpha \Rightarrow \beta$ (*c*) $L\alpha \Rightarrow M\beta$ (*d*) $\alpha \Rightarrow L\beta$ (*e*) $\alpha \Rightarrow \beta$ (*f*) $\alpha \Rightarrow M\beta$ (*g*) $M\alpha \Rightarrow L\beta$ (*h*) $M\alpha \Rightarrow \beta$ (*i*) $M\alpha \Rightarrow M\beta$ (*j*) $M(\alpha \Rightarrow \beta)$ (*k*) $L(\alpha \Rightarrow \beta)$ (*l*) $(L\alpha \Rightarrow L\beta) \wedge (M\alpha \Rightarrow M\beta)$ (*m*) $((L\alpha \Rightarrow LB) \wedge (M\alpha \Rightarrow M\beta)) \wedge ((L\beta \Rightarrow L\alpha) \wedge (M\beta \Rightarrow M\alpha))$

The interpretation of $\vdash_S F(\alpha, \beta)$ when $F(\alpha, \beta) = L\alpha \Rightarrow L\beta$ is that $L_p(v(\alpha)) \subseteq L_p(v(\beta))$ where $v(\alpha)$ is the interpretation of α in the universe U. Thus the RMP rule with (a) means:

"if α and $\beta \Rightarrow \delta$ roughly follow from Γ and the lower approximation of the interpretation of α is a subset of the lower approximation of the interpretation of β in a Kripke frame (S, ρ), then δ roughly follows from Γ".

Similarly the other cases may be interpreted.

Taking $\beta = \alpha$ the standard MP rule is obtained. Thus all the RMP rules (*a*) to (*m*) are generalization of the classical MP rule. Also it is proved that the rough consequence relation $\Vdash_R$ satisfies the Tarskian conditions of logical consequence viz.

— if sets $\alpha \in \Gamma$ then $\Gamma \Vdash_R \alpha$ (overlap or reflexivity),

— if $\Gamma, \alpha \Vdash_R \beta$ and $\Delta \Vdash_R \alpha$ then $\Gamma \cup A \Vdash_R \beta$ (cut),

— if $\Gamma \Vdash_R \alpha$ then $\gamma_1, \gamma_2, \ldots \gamma_n \Vdash_R \alpha$ for some $\gamma_1, \gamma_2, \ldots \gamma_n \in \Gamma$ (compactness),

— if $\Gamma \Vdash_R \alpha$ then if $\Gamma \cup \Delta \Vdash_R \alpha$ (dilution or monotonicity).

So, following Tarski, rough consequence logics are genuine logics and from the interpretations given above, it is clear that these bunch of rough logics can be used in approximate reasoning.

It should be mentioned that two other RMP rules are also defined in the original paper [13] which took final shapes in [4] as follows.

$$\frac{\Gamma \Vdash_R \alpha, \vdash_{S_5} M\alpha \Rightarrow M\beta}{\Gamma \Vdash_R M\alpha \wedge M\beta}$$

and

$$\frac{\Gamma \Vdash_R M\alpha, \Gamma \Vdash_R M\beta}{\Gamma \Vdash_R M\alpha \wedge M\beta}.$$

This rough consequence logic grafted on S_5 with the above two rules, captures the notion of rough truth proposed by Pawlak in [27]. This logic also turns out to be equivalent to Jaskowski's discussive logic J proposed in 1948. Interestingly, the logic system J is considered to be the predecessor of paraconsistent logic which is now a days an important branch of research.

4. MEMBERSHIP FUNCTION BASED MF-ROUGH SETS

In this section we focus on the following definition of Pawlakian rough set.

Definition 36. *A rough set is a triple* $\langle U, R, [\cdot]_{\approx} \rangle$ *where U is a nonempty set, R is an equivalence relation on U and* $[\cdot]_{\approx}$ is *an equivalence class with respect to the relation* $\approx$ *of rough equality on the power set* 2^U *of U viz.* $P \approx Q$ *if and only if* $\underline{P} = \underline{Q}$ *and* $\overline{P} = \overline{Q}$, $P, Q \subseteq U$.

4.1 Rough Membership Function

Taking the universe U as finite the notion of rough membership function was formally defined by Pawlak and Skowron in [29] and applied to develop rough mereology [30,31].

Definition 37. *Given any subset* $P \subseteq U$, *a rough membership function* f_P *is a mapping from U to Ra*[*0*, 1], *the set of rational numbers in* [0, 1], *defined by* $f_P(u) = \dfrac{Card([u]_R \cap P)}{Card([u]_R)}$ *for all* $u \in U$.

For our purpose, we take U as any set, finite or infinite, but assume that the equivalence classes $[\cdot]_R$ or blocks generated by R are all of finite cardinality.

Observation 2.

1. $f_P(u) = 1$ if and only if $u \in \underline{P}$.
2. $f_P(u) = 0$ if and only if $u \in (\overline{P})^c$.

3. $0 < f_P(u) < 1$ if and only if $u \in Bd(P) = \overline{P} - \underline{P}$.
4. $f_P(u) = f_P(v)$ for uRv.

Observation 3. Each block $[\cdot]_R$ being finite, there is a fixed set of rational numbers in [0, 1] that are admissible values for the members of the block viz. $\left\{0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}, 1\right\}$, where $Card([\cdot]_R) = n$. This set of admissible values is determined right at the beginning when the partition is formed in U. Under a rough membership function f_P all elements of a block receive the same value out of the set of admissible values associated with the particular block which will be denoted by admiss-value [·]. This value shall also be referred to as the value of the block under the rough membership function and denoted by $f_P([\cdot])$.

Observation 4. Some properties of rough membership functions are listed below.

1. If $f_P = f_Q$ then $P \approx Q$ but the converse does not hold.
2. If $P \approx Q$ then $f_P(u) = 1$ if and only if $f_Q(\mathrm{u}) = 1$ and $f_P(u) = 0$ if and only if $f_Q(u) = 0$.
3. If for some P, u, $0 < f_P(u) < 1$ then there exists $Q \neq P$ such that $f_P = f_Q$.
4. $f_{P^c}(u) = 1 - f_P(u)$ for all $u \in U$.
5. If $P \subseteq Q$ then $f_P \leq f_Q$, but the converse does not hold.
6. If $f_P \leq f_Q$ then $\underline{P} \subseteq \underline{Q}$ and $\overline{P} \subseteq \overline{Q}$, i.e., P is roughly included in Q.
7. $\max[0, f_P(u) + f_Q(u) - 1] \leq f_{P \cap Q}(u) \leq \min[f_P(u), f_Q(u)]$.
8. $\max[f_P(u), f_Q(u)] \leq f_{P \cup Q}(u) \leq \min[1, f_P(u) + f_Q(u)]$.
9. $f_{P \cup Q}(u) = f_P(u) + f_Q(u) - f_{P \cap Q}(u)$.

The results 7, 8 and 9 are proved by Yao [50].

4.2 MF-rough sets

We now give the definition of an MF-rough set.

Definition 38. [*16*] *Let $\equiv$ be the relation defined on 2^U by $P \equiv Q$ if and only if $f_P = f_Q$. Then, $\equiv$ is an equivalence relation generating a partition on 2^U. An MF-rough set is a triple $\langle U, R, [\cdot]_{\equiv} \rangle$ where U, R are as before and $[\cdot]_{\equiv}$ is a member of the quotient set $2^U/_{\equiv}$.*

Observation 5. The relation $\equiv$ generates a finer partition on 2^U than $\approx$. That is, the power set 2^U receives two partitions due to $\approx$ and $\equiv$ such that each equivalence class $[\cdot]_{\approx}$ is the union of some equivalence classes $[\cdot]_{\equiv}$.

Note 1. When U and R are fixed, any equivalence class $[\cdot]_{\approx}$ is a rough set and any equivalence class $[\cdot]_{\equiv}$ is an MF-rough set.

An MF-rough set $[P]_{\equiv}$ is a rough set if and only if $[P]_{\equiv} = [P]_{\approx}$, that is, if and only if $P \approx Q$ implies $f_P = f_Q$.

Example 5. $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}, u_{11}, u_{12}\}$,

$R = \{u_1, u_2\}, \{u_3, u_4, u_5\}, \{u_6\}, \{u_7, u_8, u_9, u_{10}\}, \{u_{11}, u_{12}\}$.

Any member of $2^U/_{\approx}$ is a rough set.

Any member of $2^U/\equiv$ is an MF-rough set.

Let us take the rough set with $\{u_7, u_8, u_9, u_{10}\}$ and $\{u_{11}, u_{12}\}$ in the lower approximation and $\{u_6\}$ as the complement of the upper approximation. That is, the equivalence classes $\{u_1, u_2\}$ and $\{u_3, u_4, u_5\}$ constitute the boundary. We display all the subsets of U belonging to this equivalence class.

Lower approximation: $\{u_7, u_8, u_9, u_{10}\} \cup \{u_{11}, u_{12}\}$. Let the element taken from the boundary set $\{u_1, u_2\}$ be u_1.

Now, elements taken from the boundary set $\{u_3, u_4, u_5\}$ may be $u_3/u_4/u_5/u_3, u_4/u_3, u_5/u_4, u_5$. Let the subsets obtained be denoted by B_1, B_2, B_3, B_4, B_5 and B_6 respectively. Similarly choosing u_2 from $\{u_1, u_2\}$ we get sets $B_7, B_8, B_9, B_{10}, B_{11}$ and B_{12}.

According to our definition this rough set is the triple $\langle U, R, \{B_1, B_2, ..., B_{12}\}\rangle$. It should be noted that all the three components X, R and the collection of sets $\{B_1, B_2, ..., B_{12}\}$ are to be essentially displayed for the definition that is taken in this paper.

Now corresponding to this rough set there are two MF-rough sets *viz.* $\langle U, R, \{B_1, B_2, B_3, B_7, B_8, B_9\}\rangle$ and $\langle U, R, \{B_4, B_5, B_6, B_{10}, B_{11}, B_{12}\}\rangle$ since f_{B_i}, $i = 1, 2, 3, 7, 8, 9$ give the same rough membership function viz. $f_{B_i}(\{u_7, u_8, u_9, u_{10}\}) = f_{B_i}(\{u_{11}, u_{12}\}) = 1$, $f_{B_i}(\{u_6\}) = 0$, $f_{B_i}(\{u_1, u_2\}) = 1/2$, $f_{B_i}(\{u_3, u_4, u_5\}) = 1/3$ and f_{B_j}, $j = 4, 5, 6, 10, 11, 12$ give the same rough membership function *viz.* $f_{B_j}(\{u_7, u_8, u_9, u_{10}\}) = f_{B_j}(\{u_{11}, u_{12}\}) = 1$, $f_{B_j}(\{u_6\}) = 0$, $f_{B_j}(\{u_1, u_2\}) = 1/2$, $f_{B_j}(\{u_3, u_4, u_5\}) = 2/3$.

From the display of R one can immediately say that $\langle U, R, \{D\}\rangle$ where D is the union of some blocks due to R is a rough set and also is an MF-rough set.

This example also shows that neither concept is a generalization of the other.

The following three fundamental theorems [16] are now stated.

Theorem 12. *Given* $\langle U, R\rangle$ *and subsets* P, Q *of* U *there exists a subset* A *of* U *such that* $f_A = f_P \wedge f_Q$ *(pointwise).*

Theorem 13. *Given* $\langle U, R\rangle$ *and subsets* P, Q *of* U *there exists a subset* B *of* U *such that* $f_B = f_P \vee f_Q$ *(pointwise).*

Theorem 14. *The subsets* A *and* B *defined in Theorems 12 and 13 satisfy the properties* $\underline{A} = \underline{P} \cap \underline{Q}$, $\overline{A} = \overline{P} \cap \overline{Q}$ and $\underline{B} = \underline{P} \cup \underline{Q}$, $\overline{B} = \overline{P} \cup \overline{Q}$.

Definition 39. *Because of Theorems 12 and 13 we are able to define rough membership function algebra (RMF-algebra) [16] for a fixed approximation space* $\langle U, R\rangle$ *viz.* $\{\{f_P\}_{P\subseteq U}, \wedge, \vee, \neg, f_\emptyset, f_U\}$ *where* $\{f_P\}_{P\subseteq U}$ *denotes the set of distinct rough membership functions not their family and* $\neg f_P = 1 - f_P = f_{P^c}$.

Theorem 15. *The RMF-algebra on an approximation space* $\langle U, R\rangle$ *is a quasi-Boolean algebra.*

We have defined two other unary operators I and C in RMF-algebra by $I f_P = f_{\underline{P}}$ and $C f_P = f_{\overline{P}}$.

These are dual operators in the sense that $\neg I \neg f_P = C f_P$ and $\neg C \neg f_P = I f_P$.

Theorem 16. *The algebra* $\{\{f_P\}_{P\subseteq U}, \wedge, \vee, \neg, \mathrm{I}, C, f_\emptyset, f_U\}$ *has the following properties:*

1. $I f_U = f_U$,
2. $I f_P < f_P$,
3. $I(f_P \wedge f_Q) = I f_P \wedge I f_Q$,
4. $I I f_P = I f_P$,
5. $C I f_P = I f_P$,
6. $\neg I f_P \vee I f_P = f_U$,
7. $I(f_P \vee f_Q) = I f_P \vee I f_Q$

Observation 6. In [1] the rough set algebra RS was proved to be a pre-rough algebra. The RMF-algebra enhanced with I and C however does not form a pre-rough algebra since it lacks the property that $If_P < If_Q$ and $Cf_P < Cf_Q$ imply $f_P < f_Q$ (see Example 6 and Note 2). This marks a significant difference between rough set structure and MF-rough set structure. In fact, the RMF-algebra is a model of abstract algebraic structure IA1+IA2 where it is unresolved till now whether an $\Rightarrow$ can be defined in general or not obeying the property ($P_{\Rightarrow}$). However, such an $\Rightarrow$ can be defined in the RMF-algebra by using Gödel arrow viz.

$$\begin{aligned} u \Rightarrow v &= 1 \textit{ if } u \leq v, \\ &= v \text{ if } u > v. \end{aligned}$$

This arrow may now be extended to rough membership functions as follows.

$$(f_P \Rightarrow f_Q)(u) = f_P(u) \Rightarrow fQ(u) \text{ for each } u.$$

So, if $f_P \leq fQ$ then $(f_P \Rightarrow f_Q)(u) = 1$ for all u, i.e., $f_P \Rightarrow f_Q = f_U$. If $f_P > f_Q$ for all u, then $(f_P \Rightarrow f_Q)(u) = f_Q(u)$, i.e., $f_P \Rightarrow f_Q = f_Q$.

If for some u, $f_P(u) < f_Q(u)$ and for some v, $f_P(v) > f_Q(v)$ then $(f_P \Rightarrow f_Q)(u) = 1$ for some u and $(f_P \Rightarrow f_Q)(v) = f_Q(v)$ for some v. In this case a subset S of U is defined by $S = \{\cup_u[u]_R : f_P(u) \leq f_Q(u)\} \cup \{\cup_v(B \cap [v]_R) : f_P(v) > f_Q(v)\}$. *So*, $f_S(u) = 1 = f_P(u) \leq f_Q(u)$ and $f_S(v) = f_Q(v) = f_P(v) \Rightarrow f_Q(v)$. Thus in any case there exists a set $S \subseteq U$ such that $f_P \Rightarrow f_Q = f_S$. So the set $\{f_P\}_{P \subseteq U}$ is closed with respect to the operation $\Rightarrow$. Clearly, $f_P \Rightarrow f_Q = f_U$ if and only if $f_P \leq f_Q$.

Two examples of RMF-algebra, one linear and other non-linear, have been presented in [35] as models of the structure IA1+ IA2. We now present here only the non-linear one.

Example 6. A eight element non-linear RMF algebra has been constructed as follows:

Let, $U = \{a, b, c, d\}$ and let the relation be $R = (\{a, b, c\} \times \{a, b, \mathrm{c}\}) \cup \{(d,d)\}$.

Now, $2^U = \{\emptyset, \{a\}, \{\mathrm{b}\}, \{\mathrm{c}\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, U\}$. The rough membership functions are defined by

$f_\emptyset(a) = 0, f_{\{a\}}(a) = 1/3, f_{\{b\}}(a) = 1/3, f_{\{c\}}(a) = 1/3, f_{\{d\}}(a) = 0$

$f_\emptyset(b) = 0, f_{\{a\}}(b) = 1/3, f_{\{b\}}(b) = 1/3, f_{\{c\}}(b) = 1/3, f_{\{d\}}(b) = 0$

$f_\emptyset(c) = 0, f_{\{a\}}(c) = 1/3, f_{\{b\}}(c) = 1/3, f_{\{c\}}(c) = 1/3, f_{\{d\}}(c) = 0$

$f_\emptyset(d) = 0, f_{\{a\}}(d) = 0, f_{\{b\}}(d) = 0, f_{\{c\}}(d) = 0, f_{\{d\}}(d) = 1$

$f_{\{a, b\}}(a) = 2/3, f_{\{b, c\}}(a) = 2/3, f_{\{c, a\}}(a) = 2/3, \mathrm{f}_{\{a, d\}}(a) = 1/3, f_{\{b, d\}}(a) = 1/3, f_{\{c, d\}}(a) = 1/3$

$f_{\{a,b\}}(b) = 2/3, f_{\{b,c\}}(b) = 2/3, f_{\{c,a\}}(b) = 2/3$, $\mathrm{f}_{\{a,d\}}(b) = 1/3, f_{\{b,d\}}(b) = 1/3, f_{\{c,d\}}(b) = 1/3$ $f_U(b) = 1$

$f_{\{a,b\}}(c) = 2/3, f_{\{b,c\}}(c) = 2/3, f_{\{c,a\}}(c) = 2/3$, $\mathrm{f}_{\{a,d\}}(c) = 1/3, f_{\{b,d\}}(c) = 1/3, f_{\{c,d\}}(c) = 1/3$ $f_U(c) = 1$

$f_{\{a,b\}}(d) = 0, f_{\{b,c\}}(d) = 0, f_{\{c,a\}}(d) = 0$, $\mathrm{f}_{\{a,d\}}(d) = 1, f_{\{b,d\}}(d) = 1, f_{\{c,d\}}(d) = 1$

$f_{\{a,b,c\}}(a) = 1, f_{\{a,b,d\}}(a) = 2/3, f_{\{a,c,d\}}(a) = 2/3, f_{\{b,c,d\}}(a) = 2/3, f_U(a) = 1$

$f_{\{a,b,c\}}(b) = 1, f_{\{a,b,d\}}(b) = 2/3, f_{\{a,c,d\}}(b) = 2/3, f_{\{b,c,d\}}(b) = 2/3, f_U(b) = 1$

$f_{\{a,b,c\}}(c) = 1, f_{\{a,b,d\}}(c) = 2/3, f_{\{a,c,d\}}(c) = 2/3, f_{\{b,c,d\}}(c) = 2/3, f_U(c) = 1$

$f_{\{a,b,c\}}(d) = 0, f_{\{a,b,d\}}(d) = 1, f_{\{a,c,d\}}(d) = 1, f_{\{b,c,d\}}(d) = 1, f_U(d) = 1$

We have, here, eight distinct rough membership functions

$\{f_\emptyset, f_{\{a\}}, f_{\{d\}}, f_{\{a,b\}}, f_{\{a,d\}}, f_{\{a,b,c\}}, f_{\{a,b,d\}}, f_U\}$.

It can be easily verified from the lattice whose Hasse diagram is shown in Figure 6 that $\{\{f_\emptyset, f_{\{a\}}, f_{\{d\}}, f_{\{a,b\}}, f_{\{a,d\}}, f_{\{a,b,c\}}, f_{\{a,b,d\}}, f_U\}, \wedge, \vee, \neg, \mathrm{I}, \mathrm{C}, f_\emptyset, f_U\}$ is a RMF-algebra which satisfies all the axioms of the structure IA1 + IA2, where $\neg$, I are given by

$\neg$	$f_\emptyset$	$f_{\{a\}}$	$f_{\{d\}}$	$f_{\{a,b\}}$	$f_{\{a,d\}}$	$f_{\{a,b,c\}}$	$f_{\{a,b,d\}}$	f_U
	f_U	$f_{\{a,b,d\}}$	$f_{\{a,b,c\}}$	$f_{\{a,d\}}$	$f_{\{a,b\}}$	$f_{\{d\}}$	$f_{\{a\}}$	$f_\emptyset$

	$f_\emptyset$	$f_{\{a\}}$	$f_{\{d\}}$	$f_{\{a,b\}}$	$f_{\{a,d\}}$	$f_{\{a,b,c\}}$	$f_{\{a,b,d\}}$	f_U
I	$f_\emptyset$	$f_\emptyset$	$f_{\{d\}}$	$f_\emptyset$	$f_{\{d\}}$	$f_{\{a,b,c\}}$	$f_{\{d\}}$	f_U
C	$f_\emptyset$	$f_{\{a,b,c\}}$	$f_{\{d\}}$	$f_{\{a,b,c\}}$	f_U	$f_{\{a,b,c\}}$	f_U	f_U

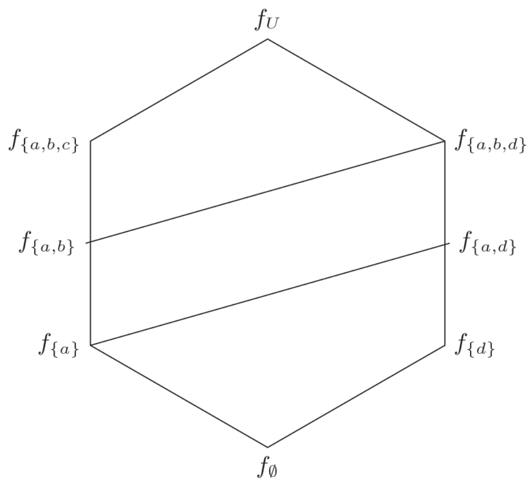

Fig. 6: Hasse diagram(non-linear RMF algebra).

Now, in this RMF-algebra the binary operation $\Rightarrow$ as mentioned above gives the following table.

$\Rightarrow$	$f_{\emptyset}$	$f_{\{a\}}$	$f_{\{d\}}$	$f_{\{a,b\}}$	$f_{\{a,d\}}$	$f_{\{a,b,c\}}$	$f_{\{a,b,d\}}$	f_U
$f_{\emptyset}$	f_U	f_U	f_U	f_U	f_U	f_U	f_U	f_U
$f_{\{a\}}$	$f_{\emptyset}$	f_U	$f_{\{d\}}$	f_U	f_U	f_U	f_U	f_U
$f_{\{d\}}$	$f_{\emptyset}$	$f_{\{a,b,c\}}$	f_U	$f_{\{a,b,c\}}$	f_U	$f_{\{a,b,c\}}$	f_U	f_U
$f_{\{a,b\}}$	$f_{\emptyset}$	$f_{\{a\}}$	$f_{\{d\}}$	f_U	$f_{\{a,d\}}$	f_U	f_U	f_U
$f_{\{a,d\}}$	$f_{\emptyset}$	$f_{\{a\}}$	$f_{\{d\}}$	$f_{\{a,b,c\}}$	f_U	$f_{\{a,b,c\}}$	f_U	f_U
$f_{\{a,b,c\}}$	$f_{\emptyset}$	$f_{\{a\}}$	$f_{\{d\}}$	$f_{\{a,b\}}$	$f_{\{a,d\}}$	f_U	$f_{\{a,b,d\}}$	f_U
$f_{\{a,b,d\}}$	$f_{\emptyset}$	$f_{\{a\}}$	$f_{\{d\}}$	$f_{\{a,b\}}$	$f_{\{a,d\}}$	$f_{\{a,b,c\}}$	f_U	f_U
f_U	$f_{\emptyset}$	$f_{\{a\}}$	$f_{\{d\}}$	$f_{\{a,b\}}$	$f_{\{a,d\}}$	$f_{\{a,b,c\}}$	$f_{\{a,b,d\}}$	f_U

Note 2. In the above example, we can see that $If_{\{a\}} = If_{\{a,b\}}$ and $Cf_{\{a\}} = Cf_{\{a,b\}}$.

So, $If_{\{a,b\}} < If_{\{a\}}$ and $Cf_{\{a,b\}} < Cf_{\{a\}}$ but $f_{\{a,b\}} \not\leq f_{\{a\}}$. Hence, this example is an instant where the property $Iu \leq Iv$ and $Cu \leq Cv$ imply $u \leq v$, for all u,v, does not hold in a RMF-algebra.

Theorem 17. *In the algebra* $(\{f_P\}_{P \subseteq U}, \wedge, \vee, \neg, I, C, f_\emptyset, f_U\}$ *let a binary relation* ρ *be defined by* $f_P \rho f_Q$ *if and only if* $I f_P = I f_Q$ *and* $C f_P = C f_Q$. *Then* ρ *is a congruence relation.*

Theorem 18. *The quotient algebra* $\langle \{f_P\}_{P\subseteq U}/\rho, \curlywedge, \curlyvee, Neg, \mathcal{I}, C, [f_\emptyset]_\rho, [f_U]_\rho\}\rangle$ *is isomorphic with the rough set algebra RS under the mapping* $\psi([f_A]_\rho) = [A]_\approx$ *where* $[f_A]_\rho \curlywedge [f_B]_\rho = [f_A \wedge f_B]_\rho = [f_P]_\rho, f_P = f_A \wedge f_B$,

$[f_A]_\rho \curlyvee [f_B]_\rho = [f_A \vee f_B]_\rho = [f_Q]_\rho, f_Q = f_A \vee f_B$,

$Neg[f_A]_\rho = [\neg f_A]_\rho$ *and* $\mathcal{I}[f_A]_\rho = [I f_A]_\rho$ *equivalently* $(\mathcal{C}[f_A]_\rho = [C f_A]_\rho)$.

4.3 A logic for MF-rough sets

A logic for MF-rough sets has been developed in [16] using the ⇒ (Gödel arrow extended over rough membership functions) discussed in Observation 6.

The logic $\mathcal{L}_{MF}$: The alphabet of the language consists of propositional variables: p_1, p_2, ..., connectives: $\neg$, $\wedge$, $\vee$, $\Rightarrow$, I, C [the same symbols are used abusively in the algebra and logic]. The wffs are defined in the usual fashion.

Axioms for $\mathcal{L}_{MFRS}$:

1. $\alpha \Rightarrow \neg\neg\alpha$
2. $\neg\neg\alpha \Rightarrow \alpha$
3. $\alpha \wedge \beta \Rightarrow \beta$
4. $\alpha \wedge \beta \Rightarrow \beta \wedge \alpha$
5. $\alpha \wedge (\beta \vee \gamma) \Rightarrow (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$
6. $(\alpha \wedge \beta) \vee (\alpha \wedge \gamma) \Rightarrow \alpha \wedge (\beta \vee \gamma)$
7. $I\alpha \Rightarrow \alpha$
8. $I\alpha \Rightarrow II\alpha$
9. $I\alpha \wedge I\beta \Rightarrow I(\alpha \wedge \beta)$
10. $CI\alpha \Rightarrow I\alpha$
11. $I(\alpha \Rightarrow \beta) \Rightarrow (I\alpha \Rightarrow I\beta)$
12. $\neg I\alpha \vee I\alpha$

Rules of inference:

1. $\dfrac{\alpha,\ \alpha \Rightarrow \beta}{\beta}$

2. $\dfrac{\alpha \Rightarrow \beta,\ \beta \Rightarrow \gamma}{\alpha \Rightarrow \gamma}$

3. $\dfrac{\alpha}{\beta \Rightarrow \alpha}$

4. $\dfrac{\alpha \Rightarrow \beta}{\neg\beta \Rightarrow \neg\alpha}$

5. $\dfrac{\alpha \Rightarrow \beta,\ \alpha \Rightarrow \gamma}{\alpha \Rightarrow \beta \wedge \gamma}$

6. $\dfrac{\alpha \vee \beta,\ \alpha \Rightarrow \gamma,\ \beta \Rightarrow \gamma}{\gamma}$

7. $\dfrac{\alpha \Rightarrow \beta}{I\alpha \Rightarrow I\beta}$

8. $\dfrac{\alpha}{I\alpha}$

The interpretation is given in a domain $\langle U,\ R\rangle$ which is an approximation space such that the equivalence classes of R are of finite cardinality.

The interpretation of a wff is defined by a valuation function v given by:

$v(p_i)$ is an arbitrary MF-rough set f_P,

$v(\neg\alpha)$ is $\neg v(\alpha)$,

$v(I\alpha)$ is $Iv(\alpha)$,

$v(C\alpha)$ is $Cv(\alpha)$,

$v(\alpha \wedge \beta)$ is $v(\alpha) \wedge v(\beta)$,

$v(\alpha \vee \beta)$ is $v(\alpha) \vee v(\beta)$,

$v(\alpha \Rightarrow \beta)$ is $v(\alpha) \Rightarrow v(\beta)$ where the last $\Rightarrow$ is a Gödel arrow extended over rough membership functions.

A wff α is valid in $\langle U, R\rangle$ if and only if $v(\alpha) = f_U$ for all valuations v in $\langle U, R\rangle$ and is universally valid if and only if it is valid in all domains.

Theorem 19. [*16*] *The formal system* $\mathcal{L}_{MF}$ *is sound with respect to MF-rough set semantics i.e. if* α *is a theorem in* $\mathcal{L}_{MF}$ *then* α *is universally valid.*

Note 3. It is not claimed that this set of axioms and rules form a minimal set. Nor do we claim completeness. Modal axioms K, T, S_4, B are all present in the above logic but remember that the base logic is quasi-Boolean instead of Boolean.

In a recent paper [23] some applications of MF-rough sets have been discussed.

5. ROUGH SET MODELS OF VARIOUS ALGEBRAS

In Section 2, we have presented a number of algebras based on qBa. Amongst them, some of the algebras such as tqBa, tqBa5, IA1, IA2, IA3, SystemI algebra, SystemII algebra etc. are stronger than qBa but weaker than pre-rough algebra. The abstract pre-rough algebra has a rough set model [1] which is described at the beginning of Section 2. Now a question may be raised: how can we construct proper set theoretic rough set models of these algebras which are basically weaker than pre-rough algebra? The phrase 'proper set theoretic rough set model' means that it should be a set model and should not reduce to a pre-rough algebra. It is to be noted that for any approximation space $\langle U, R\rangle$, $\langle 2^U/\approxeq, \sqcap, \sqcup, \neg, I, [\emptyset]_{\approxeq}, [U]_{\approxeq}\rangle$ becomes a pre-rough algebra and hence it satisfies more axioms than the axioms present in the aforesaid algebras. In [43,44] the present authors have made an effort on this issue. First, we observe that if 2^U is taken in place of $2^U/\approxeq$ (that means ordinary set is considered in place of rough set $[P]_{\approxeq}$), $\langle 2^{U}, \sqcap, \sqcup\rangle$ fails to form a lattice as $P \sqcap Q \neq Q \sqcap P$. So, we cannot proceed further using these operations $\sqcap$ and $\sqcup$. On the other hand, if set theoretic intersection and union are considered in place of $\sqcap$ and $\sqcup$ respectively then $\langle 2^U, \cap, \cup, \neg, \emptyset, U\rangle$ immediately turns into a Boolean algebra instead of a quasi-Boolean algebra as $\neg P$ usually means P^c. To overcome the situation we follow Rasiowa [34]. In [34], a representation theorem for quasi-Boolean algebra was presented. We focus our attention on this representation theorem. The notions of *quasi-complementation* and *quasi-field of subsets of a set U* have been discussed in that book. Let U be a non empty set and $g : U \to U$ be an involution i.e., $g(g(u)) = u$, for all $u \in U$. Clearly, every involution g is a bijective mapping. The *quasi-complementation* $\neg$ is defined by $\neg P = U - g(P) = g(P)^c$, for each $P \subseteq U$. Then, a collection $Q(U)$ of subsets of U, containing U and closed under set-theoretical union, intersection as well as the *quasi-*

complementation $\neg$, $\langle Q(U), \cap, \cup, \neg, \emptyset, U\rangle$ is called a *quasi-field of subsets of U.* It has also been shown that quasi-field of sets are typical examples of qBa, in the sense that every quasi-Boolean algebra is isomorphic to a quasi-field of sets. In this way, for a non empty set U, $\left\langle 2^U, \cap, \cup, \neg, \emptyset, U\right\rangle$ is a qBa, where $\neg$ is the quasi-complementation, i.e, $\neg P = g(P)^c$. It is to be noted that for an arbitrary involution g on U, the quasi-complementation and complementation (set-theoretical) of a subset P of U are not the same i.e., $\neg P(= g(P)^c) \neq P^c$ (see Example 9). Due to this fact, again a problem arises to define I. If in a generalized approximation space $\langle U, \rho\rangle$, $I\,P = \underline{P}_\rho$ is taken then $C\,P = \neg I \neg P = g\left(\underline{g(P)^c}_\rho\right)^c \neq \overline{P}^\rho$ in general. In fact, $\underline{P}_\rho$ and $\overline{P}^\rho$ are dual approximations in a generalized approximation space $\langle U, \rho\rangle$ with respect to the set theoretical complementation whereas I and C are dual operations with respect to the quasi-negation $\neg$ in the algebras discussed in Section 2 . However, we have solved the issue by defining a new approximation space $\left\langle U, \rho^g\right\rangle$ from a generalized approximation space $\langle U, \rho\rangle$, U being a non empty set and ρ being an arbitrary relation on U and g, an arbitrary involution on U. Using ρ^g, a pair of lower-upper approximations has been defined to obtain proper set theoretic rough set models of some of the algebras mentioned in section 2. Moreover, a necessary and sufficient condition is obtained when lower and upper approximations $\underline{P}_\rho$ and $\overline{P}^\rho$ in a generalized approximation space $\langle U, \rho\rangle$ satisfy the notion of duality with respect to the quasi-complementation.

5.1 A g-approximation space (U, ρ^g) and inter-relations between ρ^g and ρ

The notion of *quasi-complementation* has been discussed above. Proposition 5 below describes some of its properties.

Proposition 5. [*34*] *Let* $g : U \to U$ *be an involution, i.e.,* $g(g(u)) = u$, *for all* $u \in U$. *The following results hold.*

1. *g is a bijective mapping on U.*
2. $g(g(P)) = P$, *for all* $P \subseteq U$.
3. $g(P \cup Q) = g(P) \cup g(Q)$, *for all* $P, Q \subseteq U$.
4. $g(P \cup Q) = g(P)$ n $g(Q)$, *for all* $P, Q \subseteq U$.
5. $P = g(P)^c = g(P^c)$, *for all* $P \subseteq U$.
6. $\neg\neg P = P$, *for all* $P \subseteq U$.

7. $\neg(P \cup Q) = \neg P \cup \neg Q$, *for all* $P, Q \subseteq U$.
8. $\neg(P \cup Q) = \neg P \cap \neg Q$, *for all* $P, Q \subseteq U$.

As our primary aim is to achieve a pair of lower-upper approximations so that they are dual approximations with respect to the quasi-complementation, we have defined a new approximation space [43,44] in the following way.

Let $\langle U, \rho \rangle$ be a generalised approximation space and $g : U \to U$ be an involution. A binary relation ρ^g on U is defined as follows:

$$\text{for any two elements } u \text{ and } v \text{ in } U,\ u\rho^g v \text{ if and only if } g(u)\rho g(v). \tag{1}$$

We call $\langle U, \rho^g \rangle$ a g-generalized approximation space or simply, a g-approximation space.

As g is an involution on U, ρ can be redefined with respect to ρ^g as follows:

$$\text{for any two elements } u \text{ and } v \text{ in } U,\ u\rho v \text{ if and only if } g(u)\rho^g g(v). \tag{2}$$

As proofs of the following propositions are available in [44], we only state them here.

Proposition 6. *The following statements are equivalent in a g-approximation space* $\langle U, \rho^g \rangle$.

1. $\rho^g = \rho$.
   1. $u\rho v$ *implies* $g(u)\rho g(v)$, $\forall u, v \in U$.
   2. $g(u)\rho g(v)$ *implies* $u\rho v$, $\forall u, v \in U$.
   3. $u\rho^g v$ *implies* $g(u)\rho^g g(v)$, $\forall u, v \in U$.
   4. $g(u)\rho^g g(v)$ *implies* $u\rho^g v$, $\forall u, v \in U$.
2. $\rho \subseteq \rho^g$.
3. $\rho^g \subseteq \rho$.

Let $\rho_u = \{v \in U : u\rho v\}$ and $\rho^g_u = \{v \in U : u\rho^g v\}$. As g is an involution, it is obvious that $\rho_{g(g(u))} = \rho_u$ and $\rho^g_{g(g(u))} = \rho^g_u$, for all $u \in U$. But, there is no subset inclusion relation amongst ρ_u, $\rho_{g(u)}$, ρ^g_u and $\rho^g_{g(u)}$ in general. However, the following results show how they are related depending upon ρ and g.

Proposition 7. *The following statements are equivalent in a g-approximation space* $\langle U, \rho^g \rangle$.

1. $\rho^g_u = \rho^g_{g(u)}$ $(\rho_u = \rho_{g(u)})$, $\forall u \in U$.
2. $u\rho^g v$ $(u\rho v)$ *implies* $g(u)\rho^g v$ $(g(u)\rho v)$, $\forall u, v \in U$.
3. $g(u)\rho^g v$ $(g(u)\rho v)$ *implies* $u\rho^g v$ $(u\rho v)$, $\forall u, v \in U$.
4. $\rho^g_u \subseteq \rho^g_{g(u)}$ $(\rho_u \subseteq \rho_{g(u)})$, $\forall u \in U$.
5. $\rho^g_{g(u)} \subseteq \rho^g_u$ $(\rho_{g(u)} \subseteq \rho_u)$, $\forall u \in U$.
6. $\rho_u = \rho_{g(u)}$ $(\rho^g_u = \rho^g_{g(u)})$, $\forall u \in U$.

Proposition 8. *In a g-approximation space* $\langle U, \rho^g \rangle$, $\rho_u = g(\rho^g_{g(u)})$ *and* $\rho^g_u = g(\rho_{g(u)})$, $\forall u \in U$.

Proposition 9. [*44*] *In a g-approximation space* $\langle U, \rho^g \rangle$ *the following results hold.*

1. ρ^g *is reflexive if and only if* ρ *is reflexive.*
2. ρ^g *is symmetric if and only if* ρ *is symmetric.*
3. ρ^g *is transitive if and only if* ρ *is transitive.*
4. ρ^g *is serial if and only if* ρ *is serial.*

From the above proposition it follows that ρ^g is an equivalence relation on U if and only if ρ is so.

Proposition 10. *If* ρ^g (ρ) *is reflexive and transitive and* $\rho^g_u = \rho^g_{g(u)}$ $(\rho_u = \rho_{g(u)})$, $\forall u \in U$ *then* $\rho^g = \rho$.

Remark 4.

1. The reflexivity and transitivity of $\rho^g(\rho)$ in the above proposition are necessary. If we drop any one of them then ρ^g and ρ may not be equal. Example 7 is considered to show it.
2. Example 8 shows that the converse of the above result is not true even for an equivalence relation ρ^g.

Example 7. Let $U = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ and $g : U \to U$ be an involution defined by $g(a_1) = a_4$, $g(a_2) = a_6$, $g(a_3) = a_3$, $g(a_4) = a_4$, $g(a_5) = a_5$, $g(a_6) = a_2$.

Let $\rho = \{(a_1, a_1), (a_2, a_2), (a_3, a_3), (a_4, a_4), (a_5, a_5), (a_6, a_6), (a_1, a_4), (a_4, a_1), (a_2, a_6), (a_6, a_2), (a_3, a_5), (a_5, a_2)\}$ and $\sigma = \{(a_3, a_3), (a_5, a_5), (a_3, a_1)\}$. Then $\rho^g = \{(a_1, a_1), (a_2, a_2), (a_3, a_3), (a_4, a_4), (a_5, a_5), (a_6, a_6), (a_1, a_4), (a_4, a_1), (a_2, a_6), (a_6, a_2), (a_3, a_5), (a_5, a_6)\}$ is reflexive but not transitive and $\sigma^g = \{(a_3, a_3), (a_5, a_5), (a_3, a_4)\}$ is transitive but not reflexive. Here, $\rho^g_u = \rho^g_{g(u)}$ and $\sigma^g_u = \sigma^g_{g(u)}$, for all $u \in U$ but $\rho^g \neq \rho$ and $\sigma^g \neq \sigma$.

Example 8. U and g are the same as stated in Example 7.

Let $\rho = \{(a_1, a_1), (a_2, a_2), (a_3, a_3), (a_4, a_4), (a_5, a_5), (a_6, a_6), (a_1, a_4), (a_4, a_1)\}$. Then, $\rho^g = \rho$ and ρ^g is an equivalence relation on U but $\rho^g_{a_2} = \{u \in U : a_2 R^g u\} = \{a_2\} \neq \rho^g_{g(a_2)} = \{a_6\}$.

The quasi-complementation and set theoretic complementation of a set P, i.e., $\neg P = g(P)^c$ and P^c are not the same even when ρ is an equivalence relation, $\rho = \rho^g$ and $\rho_u = \rho_{g(u)}$, for all $u \in U$. The following example establishes this.

Example 9. The same U and g as mentioned in Example 7 have been considered for this case also.

Let $\rho = \{(a_1, a_1), (a_2, a_2), (a_3, a_3), (a_4, a_4), (a_5, a_5), (a_6, a_6), (a_1, a_4), (a_4, a_1), (a_2, a_6), (a_6, a_2)\}$. Then, $\rho^g = \rho$, ρ^g is an equivalence relation on U and $\rho_u = \rho_{g(u)}$, for all $u \in U$. Let $P = \{a_1, a_2, a_4\}$. Then, $P = g(P)^c = \{a_2, a_3, a_5\} \neq P^c = \{a_3, a_5, a_6\}$.

5.2 g-lower and g-upper approximations in a g-approximation space and rough set models of some algebras

In this subsection a pair of lower-upper approximations in the g-approximation space $\langle U, \rho^g \rangle$ will be discussed. These lower-upper approximations are dual with respect to the quasi-complementation. Their properties and rough set models of some of the algebras stated in Section 2 will be presented.

Let $\langle U, \rho^g \rangle$ be a g-approximation space and P be any subset of U. $\underline{P}_g$, the g-lower approximation of P and $\overline{P}^g$, the g-upper approximation of P, in the g-approximation space $\langle U, \rho^g \rangle$, are defined by:

$$\underline{P}_g = \{u \in U : p^g_u \subseteq P\}$$

and

$$\overline{P}^g = \{u \in U : p^g_{g(u)} \cap g(P) \neq \emptyset\}$$

Now, we present some propositions and theorems without their proofs. All of these are available in [44]. Proofs of new results, of course, are included.

Proposition 11. *$\underline{P}_g$ and $\overline{P}^g$ are dual approximations with respect to the quasi-complementation defined through g.*

Proposition 12. *$\underline{P}_g$ and $\overline{P}^g$ are respectively Pawlakian lower approximation of P in $\langle U, \rho^g \rangle$ and Pawlakian upper approximation of P in $\langle U, \rho \rangle$.*

Remark 5. It is noticeable from Example 13 that $\underline{P}_g \neq \underline{P}_\rho$ and $\overline{P}^g \neq \overline{P}^{\rho^g}$, even when ρ is an equivalence relation on *U*. Hence, for a subset *P* of *U*, $\langle \underline{P}_g, \overline{P}^g \rangle$ is different from $\langle \underline{P}_\rho, \overline{P}^\rho \rangle$ and $\langle \underline{P}_{\rho^g}, \overline{P}^{\rho^g} \rangle$. In fact, $\underline{P}_g$ is Pawlakian lower approximation of P in $\langle U, \rho^g \rangle$ and $\overline{P}^g$ is Pawlakian upper approximation of P in $\langle U, \rho \rangle$.

Note 4. In Proposition 12 we see that $\overline{P}^g = \overline{P}^\rho$. On the other hand, one may define $\underline{P}_g$ as Pawlakian lower approximation of P in $\langle U, \rho \rangle$, i.e., $\underline{P}_g = \underline{P}_\rho$. Then $\overline{P}^g$ (considering dual with respect to the quasi-complementation ¬) must be Pawlakian upper approximation of P in $\langle \qquad \rangle$, i.e., $\overline{P}^g = \overline{P}^{\rho^g}$.

It has been mentioned earlier that $\underline{P}_g$ and $\overline{P}^g$ are dual approximations with respect to the quasi-complementation. But $\underline{P}_\rho$ and $\overline{P}^\rho$ are not so. In fact, they are dual with respect to set theoretic complementation. We have established here a necessary and sufficient condition for a given involution *g* on *U*, $\underline{P}_\rho$ and $\overline{P}^\rho$ dual approximations with respect to be quasi-complementation defined through *g*.

Theorem 20. *Let $\langle U, \rho \rangle$ be a generalised approximation space and g be an involution on U. Then for any $P \subseteq U$, $\underline{P}_\rho$ and $\overline{P}^\rho$ are dual approximations with respect to quasi-complementation defined through g if and only if $\rho = \rho^g$.*

Remark 6. It is to be noted from Example 9 that the quasi-complementation and complementation of a set P i.e., $\neg P$ and P^c are not the same even when $\rho = \rho^g$. If they were the same, the above theorem would not have any significance at all. When $\rho = \rho^g$, $\underline{P}_g = \underline{P}_\rho$ and $\overline{P}^g = \overline{P}^\rho$. Hence all the properties of lower/upper approximations with respect to ρ as well as ρ^g coincide. Yet there remains one significant point. The complementation and quasi-complementation do not coincide yet the approximation operators are dual with respect to both of them.

However, when $\rho \neq \rho^g$, $\left\langle \underline{P}_g, \overline{P}^g \right\rangle \neq \left\langle \underline{P}_\rho, \overline{P}^\rho \right\rangle$ and $\left\langle \underline{P}_g, \overline{P}^g \right\rangle \neq \left\langle \underline{P}_{\rho^g}, \overline{P}^{\rho^g} \right\rangle$, still the following results hold.

Proposition 13. *In a g-approximation space* $\left\langle U, \rho^g \right\rangle$, *the following results hold.*

1. $\underline{U}_g = U$ *and* $\overline{\emptyset}^g = \emptyset$.
2. If $P \subseteq Q \subseteq U$ *then* $\underline{P}_g \subseteq \underline{Q}_g$ *and* $\overline{P}^g \subseteq \overline{Q}^g$.
3. $\underline{P \cap Q}_g = \underline{P}_g \cap \underline{Q}_g$ *and* $\overline{P \cup Q}^g = \overline{P}^g \cup \overline{Q}^g$, *for all* $P, Q \subseteq U$.

The counterpart of the modal axiom K in the form $\underline{\neg P \cup Q}_g \subseteq \neg\left(\underline{P}_g\right) \cup \underline{Q}_g$ does not hold in general. The following example shows this.

Example 10. The same U and g in Example 7 are considered for this example. Let $\rho = \{(a_1, a_1), (a_1, a_4), (a_4, a_1), (a_3, a_6)\}$. Then, $\rho^g = \{(a_4, a_4), (a_4, a_1), (a_1, a_4), (a_3, a_2)\}$. Let $P = \{a_2, a_3, a_4\}$ and $Q = \{a_5, a_6\}$. Then, $\underline{\neg p \cup Q}_g = \{a_1, a_2, a_3, a_5, a_6\}$ and $\neg\left(\underline{P}_g\right) \cup \underline{Q}_g = \{a_1, a_2, a_5, a_6\}$. Thus, $\underline{\neg P \cup Q}_g \not\subseteq \neg\left(\underline{P}_g\right) \cup \underline{Q}_g$.

Proposition 14. *A sufficient condition so that* $\underline{\neg P \cup Q}_g \subseteq \neg\left(\underline{P}_g\right) \cup \underline{Q}_g$ *holds for all* P, $Q \subseteq U$ *is that* $\rho = \rho^g$.

Proof. Let $u \in \underline{\neg P \cup Q}_g$. Then, $\rho^g_u \subseteq g(P)^c \cup Q$. Two possible cases are:

1. $u \subseteq g\left(\underline{P}_g\right)$
2. $u \notin g\left(\underline{P}_g\right)$

For the second case, it is obvious that $u \in \left(g\left(\underline{P}_g\right)\right)^c$ and hence $u \in \neg\left(\underline{P}_g\right) \cup \underline{Q}_g$. For the first case, u = $g(v)$ where $v \in \underline{P}_g$. Then, $\rho^g_v \subseteq$ P and hence $g(\rho^g_v) \subseteq g(P)$. Then by Proposition 8, $\rho_u \subseteq g(P)$ [as $u = g(v)$]. Since $\rho = \rho^g$, so, $\rho_u = \rho^g_u$ and therefore $\rho^g_u \subseteq g(P)$. This gives, $\rho^g_u \cap g(P)^c = \emptyset$. As $\rho^g_u \subseteq g(P)^c \cup$ Q so, $\rho^g_u \subseteq$ Q and therefore $u \in \underline{Q}_g \subseteq \neg\left(\underline{P}_g\right) \cup \underline{Q}_g$. Hence the result follows.

Remark 7.

1. When $\rho = \rho^g$, $\underline{P}_g$ and $\overline{P}^g$ become $\underline{P}_\rho$ and $\overline{P}^\rho$ respectively. Then $\underline{\neg P \cup Q}_g \subseteq \neg\left(\underline{P}_g\right) \cup \underline{Q}_g$ turns into $\underline{\neg P \cup Q}_\rho \subseteq \neg\left(\underline{P}_\rho\right) \cup \underline{Q}_\rho$, not identical with $\underline{P^c \cup Q}_\rho \subseteq \left(\underline{P}_\rho\right)^c \cup \underline{Q}_\rho$ (the counterpart of the modal axiom K in Boolean base)
2. Whether $\rho = \rho^g$ is a necessary condition or not for holding $\underline{\neg P \cup Q}_g \subseteq \neg\left(\underline{P}_g\right) \cup \underline{Q}_g$ is unsolved.

The following example shows that the counterpart of the modal axiom D: $\underline{P}_g \subseteq \overline{P}^g$ does not hold for a serial relation ρ^g.

Example 11. U and *g* are the same as stated in Example 7. Let $\rho = \{(a_1, a_2), (a_2, a_1), (a_3, a_1), (a_4, a_6), (a_5, a_5), (a_6, a_2)\}$ be a serial relation on *U*. Then, $\rho^g = \{(a_4, a_6), (a_6, a_4), (a_3, a_4), (a_1, a_2), (a_5, a_5), (a_2, a_6)\}$ is a serial relation. Let $P = \{a_2, a_4, a_5\}$. Then, $\underline{P}_g = \{a_1, a_3, \mathrm{a}_5, a_6\}$ and $\overline{P}^g = \{a_1, a_5, a_6\}$. Thus, $\underline{P}_g \not\subseteq \overline{P}^g$.

Theorem 21. *In a g-approximation space* $\left\langle U, \rho^g \right\rangle$, $\underline{P}_g \subseteq \overline{P}^g$ *holds for all* $P \subseteq U$ *if and only if* $\rho^g_u \cap \rho_u \neq \emptyset$, for all $u \in U$.

Proof. Let us assume that $\underline{P}_g \subseteq \overline{P}^g$ holds for all $P \subseteq$ U. Let u $\in U$. Then, particularly, $\underline{\rho^g_u}_g \subseteq \rho^g_u$ holds. This gives, $u \in \{v \in U : \rho_v \cap \rho^g_u = \emptyset\}$ [as $u \in \underline{\rho^g_u}_g$ and by Proposition 12] and hence $\rho^g_u \cap \rho_u \neq \emptyset$.

Conversely, let $\rho^g_u \cap \rho_u \neq \emptyset$, for all $u \in U$. Let u $\in \underline{P}_g$. This implies $\rho^g_u \subseteq P$ and hence $\rho^g_u \cap \rho_u \subseteq \rho_u \cap P$. This gives, $\rho_u \cap P \neq \emptyset$ [as $\rho^g_u \cap \rho_u \neq \emptyset$]. Hence, $u \in \overline{P}^g$.

Remark 8. $\rho_u^g \cap \rho_u \neq \emptyset$, for all u $\in U$, implies that $\rho^g(\rho)$ is a serial relation on U. But the converse is not true, i.e., there exists a serial relation $\rho^g(\rho)$ so that $\rho_u^g \cap \rho_u \neq \emptyset$, for some $u \in U$. In Example 11, $\rho_{a_2}^g = \{a_6\}$ and $\rho_{a_2} = \{a_1\}$. So, $\rho_{a_2}^g \cap \rho_{a_2} = \emptyset$. Thus, the condition in Theorem 21 is stronger than a serial relation. By the following Example 12, it is further noted that $\rho_u^g \cap \rho_u \neq \emptyset$, for all $u \in U$ does not imply $\rho = \rho^g$.

Example 12. U and *g* are the same as stated in Example 7.

Let $\rho = \{(a_1, a_2), (a_2, a_3), (a_2, a_4), (a_3, a_3), (a_4, a_6), (a_5, {}_{a5}), (a_6, a_3)\}$ be a serial relation on *U*. Then,

$\rho^g = \{(a_4, a_6), (a_6, a_3), (a_6, a_1), (a_3, a_3), (a_1, a_2), (a_5, a_5), (a_2, a_3)\}$ is a serial relation on *U*. Now, $\rho_{a_1} = \{a_2\}$, $\rho_{a_2} = \{a_3, a_4\}$, $\rho_{a_3} = \{a_3\}$, $\rho_{a_4} = \{a_6\}$, $\rho_{a_5} = \{a_5\}$, $\rho_{a_6} = \{a_3\}$ and $\rho_{a_1}^g = \{a_2\}$, $\rho_{a_2}^g = \{a_3\}$, $\rho_{a_3}^g = \{a_3\}$, $\rho_{a_4}^g = \{a_6\}$, $\rho_{a_5}^g = \{a_5\}$, $\rho_{a_6}^g = \{a_1, a_3\}$. Thus, $\rho_u^g \cap \rho_u \neq 0$, for all $u \in U$ but $\rho \neq \rho^g$.

Proposition 15. If ρ^g is *reflexive in a g-approximation space* $\langle U, \rho^g \rangle$, *the following results hold.*

1. $\overline{U}^g = U$ *and* $\underline{\emptyset}_g = \emptyset$.
2. $\underline{P}_g \subseteq P \subseteq \overline{P}^g$, *for all* $P \subseteq U$.

It is known to us that Pawlakian lower-upper approximations $\underline{P}_\rho$ and $\overline{P}^\rho$ satisfy the counterpart of the modal axiom B: $\overline{(\underline{P}_\rho)}^\rho \subseteq P$, for all P $\subseteq$ U, when ρ is a symmetric relation on U. But, $\overline{(\underline{P}_g)}^g \subseteq P$ does not hold even for an equivalence relation ρ^g on *U* (see Example 13). A necessary and sufficient condition is presented in Theorem 22 below so that the counterpart of the modal axiom B holds.

Theorem 22. *Let* ρ^g *be a symmetric relation in a g-approximation space* $\langle U, \rho^g \rangle$. *Then for any subset P of U,* $\overline{(\underline{P}_g)}^g \subseteq P$ *holds if and only if* $\rho^g = \rho$.

Proof. Let $\rho^g = R$. Then, $\langle \underline{P}_g, \overline{P}^g \rangle = \langle \underline{P}_\rho, \overline{P}^\rho \rangle$ and consequently for any subset *P of U*, $\overline{(\underline{P}_g)}^g \subseteq P$ holds as ρ is symmetric relation on U. Conversely, let us assume that $\overline{(\underline{P}_g)}^g \subseteq P$

holds, for any subset P of U. We shall show that $\rho \subseteq \rho^g$. If $\rho = \emptyset$ then $\rho^g = \emptyset$ and hence the result follows. Let $u\rho v$. Let $P = \rho^g_v$. Then, $\overline{\left(\underline{p^g_v}_g\right)}^g \subseteq p^g_v$ i.e., $\left\{z \in U : p_z \cap \underline{\rho^g_v}_g \neq \emptyset\right\} \subseteq \rho^g_v$. Now, $\underline{\rho^g_v}_g = \left\{w \in U : \rho^w_g \subseteq \rho^g_v\right\}$. This gives, $v \in \underline{\rho^g_v}_g$. As $u\rho v$ *so* $v \in \rho_u$ and hence $\rho_u \cap \underline{\rho^g_v}_g \neq \emptyset$. Then from definition of $\overline{\left(\underline{\rho^g_v}_g\right)}^g$, $u \in \overline{\left(\underline{\rho^g_v}_g\right)}^g$. As $\overline{\left(\underline{\rho^g_v}_g\right)}^g \subseteq \rho^g_v$ so u $\in \rho^g_v$. This gives, $u\rho^g v$ as ρ^g is symmetric. Thus, $\rho \subseteq \rho^g$. Using Remark 6, $\rho^g = \rho$.

Remark 9. By the above theorem it is clear that the counterpart of modal axiom B is possible with respect to g-lower and g-upper approximations only when $\rho^g = \rho$. Indeed, in that case, g-lower and g-upper approximations are the same with Pawlakian lower and upper approximations in the approximation space $\langle U, \rho\rangle$. But one gain, in this case, is that $\underline{P}_g$ and $\overline{P}^g$, i.e., $\underline{P}_\rho$ and $\overline{P}^\rho$ are dual approximations with respect to the quasi-complementation. From Example 9, it is further noted that complementation and quasi-complementation are not the same even when ρ is an equivalence relation.

Proposition 16. *If* ρ^g *is transitive in a g-approximation space* $\langle U, \rho^g\rangle$ *then for any subset P of U,* $\underline{P}_g \subseteq \underline{\left(\underline{P}_g\right)}_g$ *and* $\overline{\left(\overline{P}^g\right)}^g \subseteq \overline{P}^g$ *hold.*

The following example is considered to show that $\overline{\left(\underline{P}_g\right)}^g \subseteq \underline{P}_g$ may not hold even for an equivalence relation ρ^g in a g-approximation space $\langle U, \rho^g\rangle$.

Example 13. U and g are the same as mentioned in Example 7. Let ρ be an equivalence relation on U which partitions the set U into the subsets $\{a_2, a_3\}$, $\{a_4\}$, $\{a_1, a_5\}$, $\{a_6\}$ of U. Then, the equivalence relation ρ^g partitions the set U into the subsets $\{a_3, a_6\}$, $\{a_1\}$, $\{a_4, a_5\}$, $\{a_2\}$ of U. Let $P = \{a_1, a_3, a_6\}$. Then, $\underline{P}_g = \{a_1, a_3, a_6\}$ and $\overline{\left(\underline{P}_g\right)}^g = \{a_1, a_2, a_3, a_5, a_6\}$ and therefore $\overline{\left(\underline{P}_g\right)}^g \not\subseteq \underline{P}_g = P$. Further, we see that $\underline{P}_g = \{a_1, a_3, a_6\} \neq \underline{P}_\rho = \{a_6\}$ and $\overline{P}^g = \{a_1, a_2, a_3, a_5, a_6\} \neq \overline{P}^{\rho^g} = \{a_1, a_3, a_6\}$. It is also noticeable that $\underline{P}_\rho$ and $\overline{P}^\rho$ are not dual approximations with respect to the quasi-complementation as $\underline{(\neg P)}_\rho = \{a_1, a_5, a_6\} \neq \neg\left(\overline{P}^\rho\right) = \{a_1\}$ and $\overline{(\neg P)}^\rho = \{a_1, a_5, a_6\} \neq \neg\left(\underline{P}_\rho\right) = \{a_1, a_3, a_4, a_5, a_6\}$.

Theorem 23. *Let* ρ^g *be an equivalence relation in a g-approximation space* $\langle U, \rho^g \rangle$. *Then for any subset P of U,* $\overline{\left(\underline{P}_g\right)}^{g} \subseteq \underline{P}_g$ *holds if and only if* $\rho^g = \rho$.

The following example establishes that $\neg\left(\underline{P}_g\right) \cup \underline{P}_g = U$ may not hold even for an equivalence relation ρ^g with $\rho^g = \rho$.

Example 14. U, g and ρ are the same as stated in Example 8. Let $P = \{a_1, a_2, a_3\}$. Then $\underline{P}_g = \{a_2, a_3\}$ and hence $\neg\left(\underline{P}_g\right) \cup \underline{P}_g = \{a_1, a_2, a_3, a_4, a_5\} \neq U$.

We now state a necessary and sufficient condition so that $\neg\left(\underline{P}_g\right) \cup \underline{P}_g = U, \forall P \subseteq U$, holds.

Theorem 24. *Let* $\rho^g(\rho)$ *be an arbitrary relation in a g-approximation space* $\langle U, \rho^g \rangle$ *(generalized approximation space* $\langle U, \rho \rangle$*). Then for any subset P of U,* $\neg\left(\underline{P}_g\right) \cup \underline{P}_g = U\left(\neg\left(\underline{P}_\rho\right) \cup \underline{P}_\rho = U\right)$ *holds if and only if* $\rho^g_u = \rho^g_{g(u)}\left(\rho_u = \rho_{g(u)}\right)$, *for all* $u \in U$.

Proof. Let $\rho^g_u = \rho^g_{g(u)}$, for all $u \in U$. Let P be any subset of U and v be any element of U. If $v \in \neg(\underline{P}_g)$, the result follows. So, let $v \notin \neg(\underline{P}_g) = U - \{g(u) : \rho^g_u \subseteq P\}$. This gives, $v \in \{g(u) : \rho^g_u \subseteq P\}$. Then, $v = g(t)$ where $\rho^g_t \subseteq P$. As $\rho^g_t = \rho^g_{g(t)}$ [by the hypothesis], so $\rho^g_{g(t)} \subseteq P$ and hence $g(t) = v \in \underline{P}_g$. Thus $\neg\left(\underline{P}_g\right) \cup \underline{P}_g = U$, for any subset of P of U. Conversely, let $\neg\left(\underline{P}_g\right) \cup \underline{P}_g = U$, for any subset P of U. Let $u \in U$. It is to be shown that $\rho^g_u = \rho^g_{g(u)}$. Let $P = \rho^g_{g(u)}$. Then, $\underline{P}_g = \{v : \rho^g_v \subseteq \rho^g_{g(u)}\}$. We now claim that $u \in \underline{P}_g$. If not, $u \in (\underline{P}_g)$ as $\neg\left(\underline{P}_g\right) \cup \underline{P}_g = U$. Then, $u \notin \{g(v) : \rho^g_v \subseteq \rho^g_{g(u)}\}$. As g is bijective on U, let $u = g(z)$. Then, $\rho^g_z \not\subseteq \rho^g_{g(u)}$, i.e., $\rho^g_{g(u)} \not\subseteq \rho^g_{g(u)}$ [$z = g(u)$ follows from $u = g(z)$ as g is an involution], which is a contradiction. Thus, $u \in \underline{P}_g = \{v : \rho^g_v \subseteq \rho^g_{g(u)}\}$ and hence $\rho^g_u \subseteq \rho^g_{g(u)}$.

Using Remark 7, $\rho^g_u = \rho^g_{g(u)}$.

Similarly, the other part can be proved.

By Remark 7, $\rho^g_u = \rho^g_{g(u)}$ if and only if $\rho_u = \rho_{g(u)}$. So, the above Theorem 24 holds good for any one of the conditions $\rho^g_u = \rho^g_{g(u)}$ and $\rho_u = \rho_{g(u)}$.

It has been mentioned earlier that for any relation ρ, $\overline{P}^{\rho} = \overline{P}^{g}$ holds. As there is no fixed subset inclusion relation between $\underline{P}_{\rho}$ and $\underline{P}_{g}$ until $\rho = \rho^g$, the four possible cases (when $\underline{P}_{\rho} \neq \underline{P}_{g}$) that may occur are presented in Example 17 on page 301-302.

In order to view the important results of this section at a glance we refer to Table 4.

Rough set models for stqBa, stqBaD, stqBaT, stqBaB, tqBa, tqBa5 and IA1:

For a non empty set U, by Proposition 5, $\left\langle 2^U, \cap, \cup, \neg, \emptyset, U \right\rangle$ is a qBa, where 2^U is the power set of U, g is an involution on U and $\neg P = g(P)^c$.

Rough Set model for a stqBa: Let $\left\langle U, \rho^g \right\rangle$ be a g-approximation space. Then, $\left\langle 2^U, \cap, \cup, \neg, \emptyset, U \right\rangle$ is a qBa, where $\neg P = g(P)^c$, for all $P \in 2^U$. We now define IP, for all $P \subseteq U$ as $IP = \underline{P}_g$. Then by Proposition 5 and Proposition 13, $\left\langle 2^U, \cap, \cup, \neg, I, \emptyset, U \right\rangle$ is a stqBa.

Remark 10. The above model of stqBa is also a model for System0 algebra.

Rough Set model for a stqBaD: Let ρ^g be a relation on U so that $\rho^g_t \cap \rho_u \neq \emptyset$, for all $u \in U$. Then, by Proposition 5, Proposition 13 and Theorem 21, $\left\langle 2^U, \cap, \cup, \neg, I, \emptyset, U \right\rangle$ is a stqBaD.

Rough Set model for a stqBaT: For a reflexive relation ρ^g on U, by Proposition 5, Proposition 13 and Proposition 15, $\left\langle 2^U, \cap, \cup, \neg, I, \emptyset, U \right\rangle$ is a stqBaT.

Rough Set model for a stqBaB: For a reflexive and symmetric relation ρ^g on U with $\rho^g = \rho$, by Proposition 5, Proposition 13, Proposition 15 and Theorem 22, $\left\langle 2^U, \cap, \cup, \neg, I, \emptyset, U \right\rangle$ is a stqBaB.

Remark 11. By Proposition 14, the algebraic counterpart of the modal axiom K also holds in the above model of stqBaB as $\rho^g = \rho$. Thus, the above model is also a rough set model for stqBaB with modal axiom K (quasi-Boolean base).

Rough Set model for a tqBa: For any reflexive and transitive relation ρ^g on U, by Proposition 5, Proposition 13, Proposition 15 and Proposition 16, $\left\langle 2^U, \cap, \cup, \neg, I, \emptyset, U \right\rangle$ is a tqBa.

Rough Set model for a tqBa5: For any equivalence relation ρ^g on U with $\rho^g = \rho$, by Proposition 5, Propositions 13, Proposition 15, Proposition 16 and Theorem 23, $\langle 2^U, \cap, \cup, \neg, I, C, \emptyset, U \rangle$ is a tqBa5, where $IP = \underline{P}_g = \underline{P}_\rho$ and $CP = \overline{P}^g = \overline{P}^\rho$.

Remark 12. By Proposition 14, the algebraic counterpart of the modal axiom K also holds in the above model of tqBa5 as $\rho^g = \rho$. Thus, it is also a rough set model of tqBa5 with modal axiom K.

Rough Set model for a IA1: For any equivalence relation ρ^g on U with $\rho^g_u = \rho^g_{g(u)}$, for all $u \in U$, by Proposition 5, Proposition 10, Propositions 13, Proposition 15, Proposition 16, Theorem 23 and Theorem 24, $\langle 2^U, \cap, \cup, \neg, I, C, \emptyset, U \rangle$ is a IA1, where $IP = \underline{P} \quad \underline{P}$ and $CP = \overline{P}^g = \overline{P}^\rho$.

Remark 13. It has been shown in [43] that the algebraic counterpart of the modal axiom K holds in a IA1. In the above model of IA1, it also holds (by Proposition 10 and Proposition 14).

Table 4: Some results on the two lower-upper approximations

Nature of ρ	*Result*
ρ is arbitrary but $\rho \neq \rho^g$	(1) $\underline{P}_g$ and $\overline{P}^g$ are dual with respect to the quasi-complementation. (2) $\overline{P}^g = \overline{P}^\rho$ (3) $\underline{\neg P \cup Q}_g \not\subseteq \neg(\underline{P}_g) \cup \underline{Q}_g$ (4) $\underline{P}_\rho$ and $\overline{P}^\rho$ are not dual with respect to the quasi-complementation.
ρ is arbitrary but $\rho = \rho^g$	(1) $\underline{P}_g = \underline{P}_\rho$ (2) $\overline{P}^g = \overline{P}^\rho$ (3) $\underline{\neg P \cup Q}_g \subseteq \neg(\underline{P}_g) \cup \underline{Q}_g$ (4) $\underline{P}_\rho$ and $\overline{P}^\rho$ are always dual with respect to complementation as well as quasi-complementation.

ρ is a serial relation with $\rho_u \cap \rho^g_u = \emptyset$, for at least one $u \in U$.	(1) $\underline{P}_\rho \subseteq \overline{P}^\rho$ holds for all $P \subseteq U$ (2) $\underline{P}_g \subseteq \overline{P}^g$ does not hold for at least one $P \subseteq U$
ρ is a (serial) relation with $\rho_u \cap \rho^g_u \neq \emptyset$, for all $u \in U$.	(1) $\underline{P}_\rho \subseteq \overline{P}^\rho$ holds for all $P \subseteq U$ (2) $\underline{P}_g \subseteq \overline{P}^g$ holds for all $P \subseteq U$
ρ is reflexive but $\rho \neq \rho^g$	(1) $\underline{P}_g \subseteq P \subseteq \overline{P}^g = \overline{P}^\rho$. (2) $\underline{P}$ and $\overline{P}$ are not dual with respect to the quasi-complementation. (3) $\underline{P}_\rho \subseteq P$ but there is no fixed subset inclusion relation between $\underline{P}_\rho$ and $\underline{P}_g$. See Table 5 and Figure 7.
ρ is reflexive and $\rho = \rho^g$	(1) $\underline{P}_g = \underline{P}_\rho \subseteq P \subseteq \overline{P}^g = \overline{P}^\rho$. (2) $\underline{P}_\rho$ and $\overline{P}^\rho$ are always dual with respect to complementation as well as quasi-complementation.
ρ is symmetric but $\rho \neq \rho^g$	(1) $\overline{\left(\underline{P}_\rho\right)}^\rho \subseteq P$ (2) $\overline{\left(\underline{P}_g\right)}^g \not\subseteq P$
ρ is symmetric and $\rho = \rho^g$	(1) $\overline{\left(\underline{P}_g\right)}^g \subseteq P$
ρ is transitive	(1) $\underline{P}_\rho \subseteq \underline{\left(\underline{P}_\rho\right)}_\rho$ and $\overline{\left(\overline{P}^\rho\right)}^\rho \subseteq \overline{P}^\rho$ (2) $\underline{P}_g \subseteq \underline{\left(\underline{P}_g\right)}_g$ and $\overline{\left(\overline{P}^g\right)}^g \subseteq \overline{P}^g$

ρ is equivalence but $\rho \neq \rho^g$	(1) $\overline{\left(\underline{P}_\rho\right)}^{\rho} \subseteq \underline{P}_\rho$ (2) $\overline{\left(\underline{P}_g\right)}^{g} \not\subseteq \underline{P}_g$
ρ is equivalence and $\rho = \rho^g$	(1) $\overline{\left(\underline{P}_g\right)}^{g} \subseteq \underline{P}_g$
ρ is any relation with $\rho_u = \rho_{g(u)}$	(1) $\neg\left(\underline{P}_\rho\right) \cup \underline{P}_\rho = U$ (2) $\neg\left(\underline{P}_g\right) \cup \underline{P}_g = U$

5.3 Rough set models of some Implicative Topological quasi-Boolean algebras

In section 2 we have presented a number of Implicative Topological quasi-Boolean algebras where implication has been imposed satisfying the property ($P_{\Rightarrow}$). Besides, three intermediate properties IP1, IP2 and IP3 are included separately to IqBaO before adding the topological properties corresponding to the modal axioms T, S_4 and S_5. Hence, for a proper set theoretic rough set model of the above mentioned algebras two important steps have to be developed. First, an investigation for suitable operation that corresponds to $\Rightarrow$ is needed. Second, a pair of lower-upper approximations has to be constructed so that they are dual approximations with respect to the quasi-complementation and satisfies exactly one property of IP1, IP2 and IP3. The first has been achieved in two different ways. Boolean implication $P \Rightarrow Q (\equiv P^c \cup Q)$, in one way, serves the purpose smoothly. On the other hand, g image of Boolean implication $g(P \Rightarrow Q)(\equiv P \Rightarrow_1 Q)$ also fulfils the property ($P_{\Rightarrow}$). Thereafter, a study has been made to find some relations between them. For the second issue, a pair of lower-upper approximations, dual with respect to the quasi-complementation, has been constructed that fulfills the property IP1. Using the pair, rough set models for the chain of algebras IqBa1, IqBa1,T, IqBa1,4 and IqBa1,5 have been developed in [44].

Rough set models for IqBaO, IqBaT, IqBa4 and IqBa5

Rough Set model for IqBaO: Let $\left\langle U, \rho^g \right\rangle$ be a g-approximation space. Now, $\left\langle 2^U, \cap, \cup, \neg, \emptyset, U \right\rangle$ is a qBa, where $\neg P = g(P)^c$, for all $P \in 2^U$. We define $\Rightarrow$ in 2^U as follows

$$P \Rightarrow Q = P^c \cup Q, \text{ for all } P, Q \in 2^U.$$

Then, it is obvious that $P \Rightarrow Q = U$ if and only if $P \subseteq Q$ and consequently $\langle 2^U, \cap, \cup, \Rightarrow, \neg, \emptyset, U \rangle$ becomes a IqBa. We now define IP, for *all* $P \subseteq U$ as IP = $\underline{P}_g$. Then by Proposition 11 and Proposition 13, $\langle 2^U, \cap, \cup, \Rightarrow, \neg, I, \emptyset, U \rangle$ is a IqBaO.

Remark 14. Defining $\underline{P}_g = \underline{P}_\rho$, $\overline{P}^g = \overline{P}^{\rho^g}$ (by Note 4) it can be shown that $\langle 2^U, \cap, \cup, \Rightarrow, \neg, I_1, \emptyset, U \rangle$ becomes a different model for IqBaO with respect to I_1 where $I_1P = \underline{P}_\rho$.

Rough Set model for IqBaT: For any reflexive relation ρ^g on U, by Proposition 11, Proposition 13 and Proposition 15, $\langle 2^U, \cap, \cup, \Rightarrow, \neg, I, \emptyset, U \rangle$ is a IqBaT.

Similarly as Remark 14, for any reflexive relation ρ^g, $\langle 2^U, \cap, \cup, \Rightarrow, \neg, I_1, \emptyset, U \rangle$ becomes a different model for IqBaT with respect to I_1 where $I_1P = \underline{P}_\rho$.

Rough Set model for IqBa4: For any reflexive and transitive relation ρ^g on U, by Proposition 11, Proposition 13, Proposition 15 and Proposition 16, $\langle 2^U, \cap, \cup, \Rightarrow, \neg, I, \emptyset, U \rangle$ is a IqBa4.

Similarly as Remark 14, for any reflexive and transitive relation ρ^g, $\langle 2^U, \cap, \cup, \Rightarrow, \neg, I_1, \emptyset, U \rangle$ becomes a different model for IqBa4 with respect to I_1 where $I_1P = \underline{P}_\rho$.

Rough Set model for IqBa5: For any equivalence relation ρ^g on U with $\rho^g = \rho$, by Proposition 11, Propositions 13, Proposition 15, Proposition 16 and Theorem 23, $\langle 2^U, \cap, \cup, \Rightarrow, \neg, I, C, \emptyset, U \rangle$ is a IqBa5 where $IP = \underline{P}_g = \underline{P}_\rho$ and $CP = \overline{P}^g = \overline{P}^\rho$.

Note 5. As for any equivalence relation ρ^g on U with $\rho^g = \rho$, $I_1P = \underline{P}_\rho = \underline{P}_g$ = IP and therefore the models $\langle 2^U, \cap, \cup, \Rightarrow, \neg, I, C, \emptyset, U \rangle$ and $\langle 2^U, \cap, \cup, \Rightarrow, \neg, I_1, C, \emptyset, U \rangle$ are the same for IqBa5.

On the implications $\Rightarrow$ and $\Rightarrow_1$

We shall discuss about the implications $\Rightarrow$ and $\Rightarrow_1$. Some results on these two implications will be presented below.

Proposition 17. *Let g be an involution on a non-empty set U and* $P \Rightarrow_1 Q = g(P \Rightarrow Q)$, *where* $P \Rightarrow Q = P^c \cup Q$, *for all* $P, Q \in 2^U$. *Then* $P \Rightarrow_1 Q = \neg P \cup g(Q)$, *for all* $P, Q \in 2^U$.

Proof.

$$\begin{aligned} P \Rightarrow_1 Q &= g(P \Rightarrow Q) \\ &= g(P^c \cup Q) \\ &= g(P^c) \cup g(Q) \text{ [by Proposition 5]} \\ &= \neg P \cup g(Q) \text{ [by the definition of } \neg] \end{aligned}$$

Proposition 18. $P \Rightarrow_1 Q = U$ *if and only if* $P \subseteq Q$.

Proof.

$$\begin{aligned} P \Rightarrow_1 Q = U &\Leftrightarrow g(P \Rightarrow Q) = U \\ &\Leftrightarrow g(P \Rightarrow Q) = g(U) \text{ [as } g(U) = U] \\ &\Leftrightarrow P \Rightarrow Q = U \text{ [by Proposition 5]} \\ &\Leftrightarrow P \subseteq Q \text{ [by the property of Boolean implication]} \end{aligned}$$

Remark 15. From Proposition 18, it is clear that if $P \subseteq Q$ then $P \Rightarrow Q$ and $P \Rightarrow_1 Q$ are the same and equal to U. But, when $P \not\subseteq Q$ then $P \Rightarrow Q$ and $P \Rightarrow_1 Q$ may not be the same. Example 15 is an instant for this.

Example 15. U and g are the same as stated in Example 7. Let $P = \{a_1, a_2, a_3\}$ and $Q = \{a_2, a_4, a_5\}$. Then $P \Rightarrow Q = P^c \cup Q = \{a_2, a_4, a_5, a_6\} \neq P \Rightarrow_1 Q = g(P \Rightarrow Q) = \{a_1, a_2, a_5, a_6\}$.

Proposition 19. *Let g be an involution on a non empty set U. Then*

$$\{P \Rightarrow Q : P, Q \in U\} = \{P \Rightarrow_1 Q : P, Q \in U\}.$$

Proof: Let $P \Rightarrow Q \in \{P \Rightarrow Q : P, Q \in U\}$. Then,

$$\begin{aligned} P \Rightarrow Q &= P^c \cup Q \\ &= g(g(P))^c \cup g(g(Q)) \text{ [by Proposition 5]} \\ &= g(P) \Rightarrow_1 g(Q) \subseteq \{P \Rightarrow_1 Q : P, Q \in U\} \end{aligned}$$

[by Proposition 17 on page 299]

Thus, $\{P \Rightarrow Q : P, Q \in U\} \subseteq \{P \Rightarrow_1 Q : P, Q \in U\}$.

Similarly, it can be shown that $\{P \Rightarrow_1 Q : P, Q \in U\} = \{P \Rightarrow Q : P, Q \in U\}$ and hence $\{P \Rightarrow Q : P, Q \in U\} = \{P \Rightarrow_1 Q : P, Q \in U\}$.

If we define implication as $P \Rightarrow_1 Q = g(P \Rightarrow Q)$, for all $P, Q \in 2^U$ then $\langle 2^U, \cap, \cup, \Rightarrow_1, \neg, I, \emptyset, U \rangle$ and $\langle 2^U, \cap, \cup, \Rightarrow_1, \neg, I_1, \emptyset, U \rangle$ become different models for IqBaO/IqBaT /IqBa4/IqBa5 with respect to the implication $\Rightarrow_1$ as shown in Example 16.

Example 16. Let $U = \{a_1, a_2\}$ and g be an involution on U such that $g(a_1) = a_2$ and $g(a_2) = a_1$. Let ρ be any binary relation on U. Then, by Proposition 11, Proposition 13 and Proposition 18, $\langle 2^U, \cap, \cup, \Rightarrow, \neg, I, \emptyset, U \rangle$ and $\langle 2^U, \cap, \cup, \Rightarrow_1, \neg, I, \emptyset, U \rangle$ are two models of IqBaO. Now, the implications $\Rightarrow$ and $\Rightarrow_1$ on $P(U)$ act as follows:

$\Rightarrow$	$\emptyset$	$\{1\}$	$\{2\}$	U
$\emptyset$	U	U	U	U
$\{a_1\}$	$\{a_2\}$	U	$\{a_2\}$	U
$\{a_2\}$	$\{a_1\}$	$\{a_1\}$	U	U
U	$\emptyset$	$\{a_1\}$	$\{a_2\}$	U

$\Rightarrow_1$	$\emptyset$	$\{a_1\}$	$\{a_2\}$	U
$\emptyset$	U	U	U	U
$\{a_1\}$	$\{a_1\}$	U	$\{a_1\}$	U
$\{a_2\}$	$\{a_2\}$	$\{a_2\}$	U	U
U	$\emptyset$	$\{a_2\}$	$\{a_1\}$	U

As $\{a_1\} \Rightarrow \{a_2\} \neq \{a_1\} \Rightarrow_1 \{a_2\}$, the above two models of IqBaO are different with respect to $\Rightarrow$ and $\Rightarrow_1$.

Similarly, it can be shown that $\langle 2^U, \cap, \cup, \Rightarrow, \neg, I/I_1, \emptyset, U \rangle$ and $\langle 2^U, \cap, \cup, \Rightarrow_1, \neg, I/I_1, \emptyset, U \rangle$ are two different models of IqBaT//IqBa4/IqBa5 with respect to $\Rightarrow$ and $\Rightarrow_1$.

Rough set models for IqBa1, IqBa1,T, IqBa1,4 and IqBa1,5: a new pair of approximations

It is observed from Example 14 that $\neg(\underline{P}_g) \cup \underline{P}_g \neq U$ where $P = \{a_1, a_2, a_3\}$, ρ^g is an equivalence relation on U with $\rho^g = \rho$. Thus $\underline{P}_g$ and $\overline{P}^g$ do not fit with IP1. We have defined in [44] a new pair of lower and upper approximations so that it fulfils IP1. Rough set models for IqBa1, IqBa1,T, IqBa1,4 and IqBa1,5 have been constructed using these lower-upper approximations.

Let $\langle U, \rho^g \rangle$ be a g-approximation space and P be any subset of U. $\underline{P}_{g,1}$, the g, 1-lower approximation of P and $\overline{P}^{g,1}$, the g, 1-upper approximation of P, in the g-approximation space $\langle U, \rho^g \rangle$, are defined by:

$$\underline{P}_{g,1} = \{u \in U : \rho_u^g \subseteq P\} \cap \{u \in U : \rho_{g(u)}^g \subseteq P\}$$

and

$$\overline{P}^{g,1} = \{u \in U : \rho_{g(u)}^g \cap g(P) \neq \emptyset\} \cup \{u \in U : \rho_u^g \cap g(P) \neq \emptyset\}.$$

The following results are available in [44]. Without proofs they are presented below.

Proposition 20. *$\underline{P}_{g,1}$ and $\overline{P}^{g,1}$ are dual approximations with respect to the quasi-complementation $\neg$.*

Proposition 21. *$\underline{P}_{g,1}$ and $\overline{P}^{g,1}$ are respectively $\underline{P}_g \cap g(\underline{P}_g)$ and $\overline{P}^g \cup g(\overline{P}^g)$.*

Remark 16. For an arbitrary relation ρ^g, it follows from Proposition 21 and Proposition 12 that $\underline{P}_{g,1} \subseteq \underline{P}_g$ and $\overline{P}^{\rho} = \overline{P}^g \subseteq \overline{P}^{g,1}$, for all $P \subseteq U$.

Proposition 22. *If $\rho_u^g = \rho_{g(u)}^g$, for all $u \in U$ in a g-approximation space $\langle U, \rho^g \rangle$ then $\underline{P}_{g,1} = \underline{P}_g$ and $\overline{P}^{g,1} = \overline{P}^g$ for all $P \subseteq U$.*

In the following example we have shown how the three pairs $\langle \underline{P}_\rho, \overline{P}^\rho \rangle$, $\langle \underline{P}_g, \overline{P}^g \rangle$ and $\langle \underline{P}_{g,1}, \overline{P}^{g,1} \rangle$ of a particular set look like when ρ is an equivalence relation, $\rho \neq \rho^g$ and $\rho_u \neq \rho_{g(u)}$, for at least one $u \in U$.

Example 17. U, g and ρ are the same as stated in Example 7. The possible situations are presented in Table 5.

Table 5: Three lower-upper approximations of a particular set

	Case (*i*)	Case (*ii*)	Case (*iii*)	Case (*iv*)
P	$\{a_2, a_4\}$	$\{a_1, a_2, a_5\}$	$\{a_1, a_3, a_5\}$	$\{a_1, a_3, a_6\}$
$\left\langle \underline{P}_\rho, \overline{P}^\rho \right\rangle$	$\{a_4\}, \{a_2, a_3, a_4\}$	$\{a_1, a_5\}$, $\{a_1, a_2, a_3, a_5\}$	$\{a_1, a_5\}$, $\{a_1, a_2, a_3, a_5\}$	$\{a_6\}$, $\{a_1, a_2, a_3, a_5, a_6\}$
$\left\langle \underline{P}_g, \overline{P}^g \right\rangle$	$\{a_2\}, \{a_2, a_3, a_4\}$	$\{a_1, a_2\}$, $\{a_1, a_2, a_3, a_5\}$	$\{a_1\}$, $\{a_1, a_2, a_3, a_5\}$	$\{a_1, a_3, a_6\}$ $\{a_1, a_2, a_3, a_5, a_6\}$
$\left\langle \underline{P}_{g,1}, \overline{P}^{g,1} \right\rangle$	$\emptyset$, $\{a_1, a_2, a_3, a_4, a_6\}$	$\emptyset, U$	$\emptyset, U$	$\{a_3\}, U$
Remark	$\underline{P}_\rho$ and $\underline{P}_g$ have no common intersection	$\underline{P}_\rho$ and $\underline{P}_g$ have a non-void intersection.	$\underline{P}_g$ is a proper subset of $\underline{P}_\rho$	$\underline{P}_\rho$ is a proper subset of $\underline{P}_g$

It has been stated earlier that for any relation ρ, $\underline{P}_{g,1} \subseteq \underline{P}_g$ and $\overline{P}^\rho = \overline{P}^g \subseteq \overline{P}^{g,1}$ hold. As there is no fixed subset inclusion relation between $\underline{P}_\rho$ and $\underline{P}_g$ until $\rho = \rho^g$, the four circumstances that we have shown in Table 5 are the only possible cases when $\underline{P}_\rho \neq \underline{P}_g$. A Pictorial representation of these four situations are shown in more general way in Figure 7.

Proposition 23. *In a g-approximation space* $\langle U, \rho^g \rangle$, *the following results hold.*

1. $\underline{U}_{g,1} = U$ *and* $\overline{\emptyset}^{g,1} = \emptyset$.
2. *If* $P \subseteq Q \subseteq U$ *then* $\underline{P}_{g,1} \subseteq \underline{Q}_{g,1}$ *and* $\overline{P}^{g,1} \subseteq \overline{Q}^{g,1}$.
3. $\underline{P \cap Q}_{g,1} = \underline{P}_{g,1} \cap \underline{Q}_{g,1}$ *and* $\overline{P \cup Q}^{g,1} = \overline{P}^{g,1} \subseteq \overline{Q}^{g,1}$, *for all* $P, Q \subseteq U$.
4. $\neg \left(\underline{P}_{g,1}\right) \cup \underline{P}_{g,1}$, *for all* $P \subseteq U$.

Proposition 24. If ρ^g is *reflexive in a g-approximation space* $\langle U, \rho^g \rangle$, *the following results hold.*

1. $\overline{U}^{g,1} = U$ *and* $\underline{\emptyset}_{g,1} = \emptyset$.
2. $\underline{P}_{g,1} \subseteq P \subseteq \overline{P}^{g,1}$, *for all* $P \subseteq U$.

If ρ^g is transitive, even an equivalence relation, then $\underline{P}_{g,1} \subseteq \underline{(\underline{P}_{g,1})}_{g,1}$ may not hold. The example given below is one such.

Example 18. U and g are the same as stated in Example 7. Let $\rho = \{(a_1, a_1), (a_2, a_2), (a_3, a_3), (a_4, a_4), (a_5, a_5), (a_6, a_6), (a_1, a_2), (a_2, a_1)\}$. Then, $\rho^g = \{(a_1, a_1),(a_2, a_2), (a_3, a_3), (a_4, a_4), (a_5, a_5), (a_6, a_6), (a_4, a_6), (a_6, a_4)\}$ is an equivalence relation on U. Let $P = \{a_1, a_4, a_6\}$. Then, $\underline{P}_{g,1} = \{a_1, a_4\}$ but $\underline{(\underline{P}_{g,1})}_{g,1} = \emptyset$.

Proposition 25. If ρ^g *is transitive and* $\rho^g_u = \rho^g_{g(u)}$, *for all* $u \in U$ *in a g-approximation space* $\langle U, \rho^g \rangle$ *then for any subset P of U,* $\underline{P}_{g,1} \subseteq \underline{(\underline{P}_{g,1})}_{g,1}$ *and* $\overline{(\overline{P}^{g,1})}^{g,1} \subseteq \overline{P}^{g,1}$ *hold.*

Remark 17. The condition as stated in the above proposition is a sufficient condition but not necessary. The following example establishes that for any subset P of U, $\subseteq \underline{(\underline{P}_{g,1})}_{g,1}$ holds where ρ^g is transitive, even an equivalence relation, but $\rho^g_u \neq \rho^g_{g(u)}$, for all $u \in U$.

Example 19. Let $U = \{a_1, a_2\}$ and $g : U \to U$ be an involution defined by $g(a_1) = a_2$, $g(a_2) = a_1$. Let $\rho = \{(a_1, a_1), (a_2, a_2)\}$. Then $\rho^g = \{(a_2, a_2), (a_1, a_1)\}$ is an equivalence relation on U. Here, $\underline{P}_{g,1} = \underline{(\underline{P}_{g,1})}_{g,1}$, for all subset P of U but $\rho^g_{a_1} \neq \rho^g_{g(a_1)}$ and $\rho^g_{a_2} \neq \rho^g_{g(a_2)}$.

Proposition 26. If ρ^g *is an equivalence relation and* $\rho^g_u = \rho^g_{g(u)}$, *for* $u \in U$ *in a g-approximation space* $\langle U, \rho^g \rangle$ *then for any subset P of U,* $\overline{(\underline{P}_{g,1})}^{g,1} \subseteq \underline{P}_{g,1}$.

The condition as stated in the above proposition is only sufficient. The example given below shows that for any subset P of U, $\overline{(\underline{P}_{g,1})}^{g,1} \subseteq \underline{P}_{g,1}$ holds for an equivalence relation ρ^g where $\rho^g_u = \rho^g_{g(u)}$, for all $u \in U$.

Example 20. U, g and ρ are the same as stated in Example 19. Here, $\overline{(\underline{P}_{g,1})}^{g,1} \subseteq \underline{P}_{g,1}$, for all subset P of U but $\rho^g_{a_1} \neq \rho^g_{g(a_1)}$ and $\rho^g_{a_2} \neq \rho^g_{g(a_2)}$.

Rough Set model for IqBa1: Let $\langle U, \rho^g \rangle$ be a g-approximation space. By Proposition 20 and Proposition 23, $\langle 2^U, \cap, \cup, \Rightarrow, \neg, I, \emptyset, U \rangle$ is a IqBa1 where $\neg P = g(P)^c$, $P \Rightarrow Q = P^c \cup Q$ and $IP = \underline{P}_{g,1}$.

Rough Set model for IqBa1,T: For any reflexive relation ρ^g on U, by Proposition 20, Proposition 23 and Proposition 24, $\left\langle 2^U, \cap, \cup, \Rightarrow, \neg, I, \emptyset, U \right\rangle$ is a IqBa1,T.

Rough Set model for IqBa1,4: For any reflexive and transitive relation ρ^g with $\rho^g_u = \rho^g_{g(u)}$, for all $u \in U$, by Proposition 20, Proposition 23, Proposition 24 and Proposition 25, $\left\langle 2^U, \cap, \cup, \Rightarrow, \neg, I, \emptyset, U \right\rangle$ is a IqBa1,4 where $IP = \underline{P}_{g,1} = \underline{P}_g = \underline{P}_\rho$ by Proposition 9.

Rough Set model for IqBa1,5: For any equivalence relation ρ^g on U with $\rho^g_u = \rho^g_{g(u)}$, for all $u \in U$, by Proposition 20, Proposition 23, Proposition 24, Proposition 25 and Proposition 26, $\left\langle 2^U, \cap, \cup, \Rightarrow, \neg, I, \emptyset, U \right\rangle$ is a IqBa1,5 where $IP = \underline{P}_{g,1} = \underline{P}_\rho$ and $CP = \overline{P}^{g,1} = \overline{P}^\rho$.

Remark 18. $\left\langle 2^U, \cap, \cup, \Rightarrow_1, \neg, I, \emptyset, U \right\rangle$ becomes a different model of IqBaO/IqBaT/IqBa4/IqBa5 with respect to the implication $\Rightarrow_1$, where $P \Rightarrow_1 Q = g(P \Rightarrow Q)$ for all $P, Q \subseteq U$.

In order to view the important results of this section at a glance we refer to Table 6.

Table 6: Some important results on the three lower-upper approximations

Nature of ρ	*Result*
ρ is arbitrary relation	$\underline{P}_g$, $\overline{P}^g$ and $\underline{P}_{g,1}$, $\overline{P}^{g,1}$ are always dual approximations with respect to the quasi-complementation.
ρ is arbitrary, $\rho \neq \rho^g$ and $\rho_u = \rho_{g(u)}$, for at least one $u \in U$	(1) $\underline{P}_{g,1} \subseteq \underline{P}_g$. (2) $\overline{P}^\rho = \overline{P}^g \subseteq \overline{P}^{g,1}$. (3) $\underline{P}_\rho$ and $\overline{P}^\rho$ are not dual approximations with respect to the quasi-complementation.
ρ is reflexive/equivalence, $\rho \neq \rho^g$ and $\rho_u \neq \rho_{g(u)}$, for at least one $u \in U$	(1) $\underline{P}_{g,1} \subseteq \underline{P}_g \subseteq P \subseteq \overline{P}^g = \overline{P}^\rho \subseteq \overline{P}^{g,1}$ (2) $\underline{P}_\rho$ and $\overline{P}^\rho$ are not dual approximations with respect to the quasi-complementation. (3) $\underline{P}_\rho \subseteq P$ but there is no fixed subset inclusion relation between _ and $\underline{P}_g$. See Table 5 and Figure 7.

ρ is reflexive and transitive, $\rho_u = \rho_{g(u)}$, for all $u \in U$ and $\rho \neq \rho^g$	The case is not possible by Proposition 9.
ρ is arbitrary but not reflexive and transitive, $\rho_u = \rho_{g(u)}$, for all $u \in U$ and $\rho \neq \rho^g$	(1) $\underline{P}_{g,1} = \underline{P}_g$. (2) $\overline{P} = \overline{P}^g = \overline{P}^{g,1}$. (3) $\underline{P}_\rho$ and $\overline{P}^\rho$ are not dual approximations with respect to the quasi-complementation.
ρ is arbitrary, $\rho = \rho^g$ and $\rho_u \neq \rho_{g(u)}$, for at least one $u \in U$	(1) $\underline{P}_{g,1} \subseteq \underline{P}_g = \underline{P}_\rho$ (2) $\overline{P}^\rho = \overline{P}^g \subseteq \overline{P}^{g,1}$. (3) $\underline{P}_\rho$ and $\overline{P}^\rho$ are dual approximations with respect to the quasi-complementation.
ρ is reflexive/equivalence, $\rho = \rho^g$ and $\rho_u \neq \rho_{g(u)}$, for at least one $u \in U$	(1) $\underline{P}_{g,1} \subseteq \underline{P}_g = \underline{P}_\rho \subseteq P \subseteq \overline{P}^\rho = \overline{P}^g \subseteq \overline{P}^{g,1}$ (2) $\underline{P}_\rho$ and $\overline{P}^\rho$ are dual approximations with respect to the quasi-complementation.
ρ is arbitrary, $\rho = \rho^g$ and $\rho_u = \rho_{g(u)}$, for all $u \in U$	(1) $\underline{P}_{g,1} = \underline{P}_g = \underline{P}_\rho$ (2) $\overline{P}^\rho = \overline{P}^g = \overline{P}^{g,1}$ (3) $\underline{P}_\rho$ and $\overline{P}^\rho$ are dual approximations with respect to the quasi-complementation.

6. CONCLUDING REMARKS

We have discussed various abstract algebraic structures emerging out of various kinds of rough sets starting from the Pawlakian one. But there has been a number of such algebras all coming out of the basic one viz. topological quasi-Boolean algebra. From the angle of application these abstract algebras need to have set-models. It has been possible to present set-models to some (but not all) of these abstract structures. So this part of the study remains incomplete.

Logics corresponding to these algebras have been studied extensively. However, logics are of three kinds: first, those in which a formula is evaluated as an element of the algebra belonging to a class, second where formulas are interpreted as subsets of a universe endowed

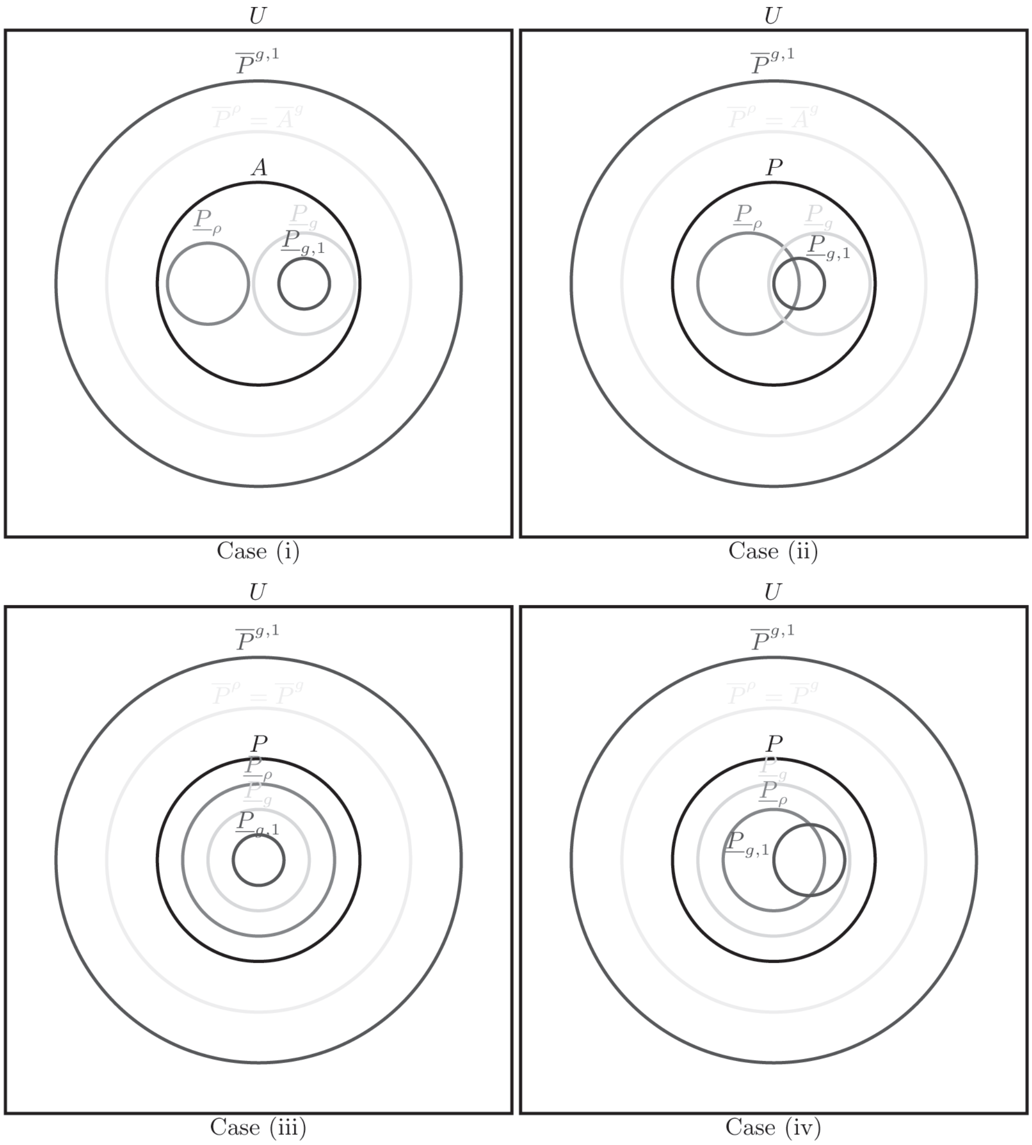

Fig. 7: Different possibilities of three lower-upper approximations when ρ is reflexive/ equivalence, $\rho \neq \rho^g$ and $\rho_u \neq \rho_{g(u)}$, for at least one $u \in U$

with topological operators on the subsets of it. In the second case, the topological operators are interpretation of the modal operators, they represent the lower/upper approximations of the sets. A third kind of logic also exists—logics with rough consequence, in which the consequence relation has been generalized via generalization of Modus Ponens rule.

Lots of question remain unanswered. Of them a few important ones are the following.

1. In Table 2, there are covering systems whose corresponding logical systems have yet not been obtained. Particularly significant will be those in which duality of the lower upper approximation does not hold and those which are non-normal in the sense that the axiom K does not hold in them. But they are rough set models in the sense that approximation operators are available in them defined in terms of covering of the universe of discourse.
2. In Section 5, a few rough set models have been constructed from the point of relational approach. A new approximation space has been defined in order to obtain lower-upper approximations to be dual with respect to quasi-complementation. A collection of relations $\{\rho : \rho = \rho^g\}$ has been identified so that Pawlakian lower-upper approximations in these approximation spaces $\langle U, \rho\rangle$ are dual with respect to the complementation as well as quasi-complementation. In other words, we obtain a Boolean based and, at the same time, a quasi-Boolean based algebra. This observation may open up a study in the field of rough set theory. Besides, another investigation may be made on covering cases. Various lower-upper approximations based on covering are available in many literature. Some of them are dual with respect to set-complementation whereas others are not so. An attempt may be taken in favour of capturing the notion of duality with respect to quasi-complementation in these lower-upper approximations. This may lead us to construct rough set model of remaining algebras discussed in Section 2. Moreover, it may give a new direction of research regarding complementation and quasi-complementation in covering based rough set theory. However, the second author of this paper has taken an initiative [56] in this direction.
3. Another interesting as well as important issue is raised below. In all set-models approximation operators are defined in terms of granules of the universe. The basic philosophy of rough set study is that the universe of discourse is granulated, elements or objects within the same granule are indiscernible. Granules are in a sense the atoms of the universe. Now the question is, what should the basic properties of the granules?

A few attempts in this direction have been made so far [15,3]. An incomplete study in this respect was presented in International Joint Conference on Rough Sets, 3-7 July 2017 Olsztyn, Poland. We present below some snapshots from that lecture to the consideration of the readers.

What is done with these granules?

— Ultimately approximating a subset of the universe in terms of subsets formed out of the granules.

— Presented philosophically, a concept is thus understood/described in terms of two better understood concepts. For example, the definable sets in the Pawlakian case.

— The rudimentary or atomic concepts are represented (extensionally) by the granules.

— A demonstrable concept is one whose extension has the same lower and upper approximations. These may be considered as the most understood concepts i.e. without any ambiguity—expectedly $\underline{G} = \overline{G}$ for any granule *G*.

— The next purpose is to understand complex concepts like *P* & *Q*, *P* or *Q*, non-*P* etc. by approximations again.

From the angle of application granules are tangible and useful clusters of points. Research in rough sets leads us to the need for developing a proper theory of granulation so as to be able to address the fundamental issues of axiomatizing as well as capturing the requirements of application satisfactorily.

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